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O. Neugebauer **M. H. Stone** **O. Veblen**

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Mathematical Reviews

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FOUNDATIONS

*Hilbert, D., und Ackermann, W. **Grundzüge der theoretischen Logik.** Dover Publications, New York, N. Y., 1946. viii+133 pp. $2.50.

Photographic reprint of the second edition [Springer, Berlin, 1938].

Post, Emil L. A variant of a recursively unsolvable problem. Bull. Amer. Math. Soc. 52, 264–268 (1946). [MF 16189]

This paper presents a very simple and interesting example of a problem which is unsolvable by any recursive (effective) means. This correspondence decision problem, as Post calls it, is: given a finite list of couples (g_1, g_1'), (g_2, g_2'), $\cdots$, (g_u, g_u'), where each g_i is a finite sequence of terms containing only a and b (as, for example, *aaabbabbaa*), to determine values of n and of $i_1, i_2, \cdots, i_n$ such that

$$g_{i_1}g_{i_2}\cdots g_{i_n}=g_{i_1}'g_{i_2}'\cdots g_{i_n}'$$

for $n\geqq 1$ and $1\leqq i_j\leqq u$. In certain cases this problem may readily be solved, as, for example, where $u=3$, $g_1=bb$, $g_2=ab$, $g_3=b=g_1'$, $g_2'=ba$, $g_3'=bb$, $i_1=1$, $i_2=i_3=2$, and $i_4=3$. The proof of the unsolvability of this problem in its full generality consists essentially of a reduction of the decision problem for the class of normal systems on a, b [see the same Bull. 50, 284–316 (1944); these Rev. 6, 29], which is known to be recursively unsolvable, to this correspondence decision problem. *R. M. Martin* (Bryn Mawr, Pa.).

Griss, G. F. C. Negationless intuitionistic mathematics. Nederl. Akad. Wetensch. Verslagen, Afd. Natuurkunde 53, 261–268 (1944). (Dutch. German, English and French summaries) [MF 15787]

In order to eliminate the idea of negation from mathematics, the author investigates how the relation of difference (nonidentity) can be replaced by a positively defined relation of distinctness. If, for the elements of a certain species S (in the sense of Brouwer [Math. Ann. 93, 244–257 (1925)]), relations $A=B$ (equality) and $A\neq B$ (distinctness) have been defined, satisfying the usual axioms for equality, $A\neq B\rightarrow B\neq A$ and $A=B$, $B\neq C\rightarrow A\neq C$, we can determine for every A in S the subspecies $Q(A)$ of the elements which are equal to A. Then $A=B$ implies $Q(A)=Q(B)$ and inversely; $A\neq B$ implies $Q(A)\neq Q(B)$ and inversely. In many cases, for example, in the theory of real numbers, the relation $\neq$ possesses the further properties: $A\neq B\rightarrow$ (for every C) either $C\neq A$ or $C\neq B$; if $A\neq C$ for every C which satisfies $C\neq B$, then $A=B$. A number of proofs, taken from various branches, are so altered that negation no longer enters in them. *A. Heyting* (Amsterdam).

Heyting, A. Untersuchungen über intuitionistische Algebra. Verh. Nederl. Akad. Wetensch. Afd. Natuurk. Sect. 1. 18, no. 2, 36 pp. (1941). [MF 13711]

This paper deals from the intuitionistic standpoint with the following topics: (1) linear equations in a field, (2) divisibility and factorization of polynomials, (3) theory of elimination. Two relations $a=b$, $a\#b$ (a equals b, a is distinct ("entfernt") from b) are basic for the intuitionistic notion of a field; $a\#b$ is not the negation of $a=b$, although it is supposed that, whatever the numbers a, b, the assumption $a=b$ is incompatible with $a\#b$ and that $a=b$ holds whenever the assumption $a\#b$ leads to a contradiction. Such pairs of mutually exclusive positive concepts as $=$ and $\#$ are typical for intuitionistic algebra. Another example, on which the theory of linear equations hinges, is the linear dependence and independence of vectors, m vectors $\mathfrak{x}_1, \cdots, \mathfrak{x}_m$ being independent ("frei") over a field K if

$$c_1\mathfrak{x}_1+\cdots+c_m\mathfrak{x}_m\#0, \qquad c_i\varepsilon K,$$

whenever at least one $c_i\#0$. In the domain of polynomials the contrast between composite and prime polynomials is of a similar nature.

The rank of a system of linear equations is defined and the classical propositions are proved first without, then by means of, determinants. A polynomial $f(x)=f_0+f_1x+\cdots+f_nx^n$ over K of formal degree n is distinct from 0 if at least one of the coefficients $f_0, f_1, \cdots$ is distinct from 0; it is fluctuating if $f(x)-f_0\#0$; it is regular and n its actual degree if $f_n\#0$. The polynomial $f(x)$ is indivisible by $g(x)$ if $f-gh\#0$ for every polynomial $h(x)$ in K; a fluctuating $f(x)$ is prime provided $f-g_1g_2\#0$ for any two fluctuating g_1, g_2. The polynomials $f_1, f_2, \cdots, f_k$ are relatively prime if for every fluctuating g at least one of the f_i is indivisible by g. The ring of residue classes modulo $f(x)$ is an integrity domain whenever f is prime, and a field if f is also regular (and of degree greater than 0). Let $y_1, y_2, \cdots, y_h$ be indeterminates and $f(x), f_1(x), \cdots, f_h(x)$ relatively prime polynomials in K; is it true that $f(x)+y_1f_1(x)+\cdots+y_hf_h(x)$ is prime over $K(y_1, \cdots, y_h)$? The question is answered in the affirmative for $h=1$, but the author has not succeeded in settling it for higher h.

This forces him to limit the decisive parts of the theory of elimination to algebraically closed fields K. For a set of homogeneous polynomials f_i depending on n variables over such a field K, form the system of resultants R_μ. The following statements are true: (1) if for every vector $\xi\#0$ at least one $f_i(\xi)\#0$ then one $R_\mu\#0$, and vice versa; (2) if all $f_i(\xi)=0$ for a certain $\xi\#0$ then all $R_\mu=0$; vice versa, if all $R_\mu=0$ then the statement that the equations have no solution $\xi\#0$ is absurd (although the actual existence of a solution remains in doubt).

The (unique) factorization of polynomials cannot universally be maintained, but, given a polynomial over an algebraically closed field K, the statement that this polynomial cannot be factored into prime polynomials is absurd. This negative result is used for the demonstration of such theorems as this: let f, g, φ be polynomials over the algebraically closed K, $fg=\varphi^m$, m a natural number, φ prime; then $f=c\varphi^k$ with a certain constant c and a nonnegative integral exponent k.

The methods employed follow closely the classical pattern. However, where one has the choice between several

methods or several equivalent formulations of a method in classical algebra, often only one of these methods, after fortification by some supplementary refinements, will do in intuitionistic algebra. *H. Weyl* (Princeton, N. J.).

Kleene, S. C. On the interpretation of intuitionistic number theory. J. Symbolic Logic 10, 109–124 (1945).

This paper summarizes results of the author and David Nelson, with sketches of proofs. The object is a formal system S, which approximately represents intuitionistic number theory. An "elementary formula" contains no logical symbol and is said to be realized by 0 if it is true (that is, represents an intuitionistically true proposition). By rules like this: "the formula $A \supset B$ is realized by the Gödel number e of a partial recursive function φ such that, whenever a realizes A, then $\varphi(a)$ realizes B," realization numbers are assigned to more complex formulas. These definitions may be interpreted intuitionistically (int.) or classically (cl.); the author distinguishes also between inferences which are int. and cl. valid, respectively. Int. every provable formula is realizable. As a corollary, in a specified sense in int. number theory no other than general recursive functions are definable. To every formula A in S (possibly containing free variables) a formula $\mathfrak{R}A$ is adjoined, such that the predicates "A is realizable" and "$\mathfrak{R}A$ is true" are equivalent. The formulas $A \supset \mathfrak{R}A$ and $\mathfrak{R}A \supset A$ are unprovable in S; by annexing them to S a consistent system S' is obtained, which diverges from the classical. In S' every provable formula is realizable. Cl., a formula can be true but unrealizable, or realizable but untrue. Incidentally, a proof is given of the fact that $\neg\neg(\forall x \neg\neg A(x) \supset \neg\neg\forall x A(x))$ is unprovable in the int. predicate calculus. The author makes it plausible that int. "A is true" is equivalent to "A is realizable." Of course no definitive statement is possible on this question. *A. Heyting* (Amsterdam).

Chandrasekharan, K. A further note on intuitionistic set-theory. Math. Student 13, 49–51 (1945). [MF 15642]

The author clarifies Brouwer's notion of a set (as distinguished from a species) by means of a very precise definition, together with examples. This is an improvement on the discussion given in the author's earlier paper [Math. Student 9, 143–154 (1941); these Rev. 4, 126].

O. Frink (State College, Pa.).

Robinson, Raphael M. Finite sequences of classes. J. Symbolic Logic 10, 125–126 (1945).

In an axiomatic set theory which distinguishes sets from classes, only sets being allowed as elements, it is impossible to represent a finite sequence of classes in the usual way as the class of ordered pairs whose first members are the argument integers and whose second members are the corresponding value classes (since this would involve the value classes' being elements). The author points out that such sequences can be represented by the class of ordered pairs whose first members are the argument integers and whose second members are the elements of the corresponding value classes. A slight adjustment allows the null class to be obtained as a value. *L. H. Loomis* (Cambridge, Mass.).

***Klein, Felix. Elementary Mathematics from an Advanced Standpoint. Vol. I. Arithmetic, Algebra, Analysis.** Dover Publications, New York, N. Y., 1945. ix+274 pp. $3.50.

Photographic reprint of the translation by E. R. Hedrick and C. A. Noble [Macmillan, New York, 1932] from the third German edition [Springer, Berlin, 1924].

Bellon, Waldemar. New perspectives in modern mathematics. Univ. Nac. Colombia 5, 363–366 (1946). (Spanish) [MF 16539]

***Speiser, Andreas. Die mathematische Denkweise.** Second edition. Verlag Birkhäuser, Basel, 1945. 122 pp. (9 plates). 14.50 Swiss francs.

[The first edition was published by Rascher, Zürich-Leipzig-Stuttgart, 1932.] It is stated in the preface that "the ancient doctrine of the mathematical character of the soul (mathematische Natur der Seelenkräfte)" constitutes the guiding principle of the discussion. To determine the content of this doctrine, beyond rather vague sentences about symmetry and harmony, seems to be rather difficult, at least to the reviewer. The author states that the discussion of logic is purposely omitted. Most mathematicians would probably prefer to say that all that is specifically mathematical is omitted. It is the "mathematical" character of music and art which is the main subject of discussion, illustrated by means of chapters on Dante, the "Farbenlehre" of Goethe, astrology and Kepler, a discussion which is apparently deeply influenced by the speculations of the neo-Platonists Plotinus and Proclus. It is only fair to state explicitly that the attitude towards method and contents of a book of the present type is largely a matter of personal taste and that it may well be the reviewer's fault if he cannot see the relations to what he himself (rather vaguely) thinks to be mathematics. *O. Neugebauer.*

Greenwood, Thomas. La valeur humaniste des mathématiques. Rev. Trimest. Canad. 32, 18–31 (1946). [MF 16432]

ALGEBRA

Gleissberg, W. Eine Aufgabe der Kombinatorik und Wahrscheinlichkeitsrechnung. Rev. Fac. Sci. Univ. Istanbul (A) 10, 25–35 (1945). (German. Turkish summary) [MF 16715]

Let $N(n, k)$ be the number of permutations of numbers 1 to n with k extreme elements, an extreme element being greater than, or less than, each of its two neighbors. Then $N(n, k) = P(n, k+1)$, the number with $k+1$ sequences, or runs up and runs down. The author derives André's recurrence [E. Netto, Lehrbuch der Combinatorik, Teubner, Leipzig, 1901, p. 108] and from it a recurrence for the probability that a permutation has more than k extreme elements. A table of the last for $n=4$ to 25 is given.

J. Riordan (New York, N. Y.).

Hadwiger, H. Eine Bemerkung über zufällige Anordnungen der natürlichen Zahlen. Mitt. Verein. Schweiz. Versich.-Math. 46, 105–109 (1946). [MF 16688]

Recurrence relations and explicit formulas are given for the numbers A_n^x of permutations of numbers 1 to n with exactly x of the successions $12, 23, \cdots, (n-1)n$ and the first term of the asymptotic expansion $A_n^x \sim (x!e)^{-1}$ is found.

The author ignores Whitworth's classical result [Choice and Chance, 5th ed., Cambridge, 1901, p. 102] that $A_n^0=\Delta^n 0!+\Delta^{n-1}0!$. *J. Riordan* (New York, N. Y.).

Finney, D. J. Orthogonal partitions of the 5×5 Latin squares. Ann. Eugenics 13, 1–3 (1946). [MF 16365]

In an earlier paper [same Ann. 12, 213–219 (1945); these Rev. 7, 107] the author defined orthogonal partitions of a Latin square. He now illustrates this notion by giving a complete enumeration of such partitions of each of the two essentially different types of 5×5 square. He finds, in particular, that the second type admits three (1^5) partitions (that is, three kinds of Eulerian square), whereas the first type admits none. *H. S. M. Coxeter* (Toronto, Ont.).

Erdös, Paul, and Kaplansky, Irving. The asymptotic number of Latin rectangles. Amer. J. Math. 68, 230–236 (1946). [MF 16419]

An n by k Latin rectangle is an array of n rows and k columns, with the integers $1, \cdots, n$ in each row and distinct integers in each column. Let $f(n, k)$ denote the number of such rectangles; for example, $f(3, 3)=12$. Riordan [Amer. Math. Monthly 53, 18–20 (1946); these Rev. 7, 233] obtained an explicit formula for the integer ${}_3K_n=f(n, 3)/n!$ and deduced the asymptotic expression ${}_3K_n\sim(n!)^2e^{-3}$, which had been conjectured by Kerawala. The present authors extend this to

$$f(n, k)\sim(n!)^k e^{-k(k-1)/2},$$

not merely for any fixed value of k (as n tends to infinity) but for $k<(\log n)^{3/2-\epsilon}$. They also indicate the beginning of an asymptotic series, which gives, in particular,

$$f(n, 3)=(n!)^3e^{-3}(1-1/n-1/2n^2+\cdots).$$

The first three terms provide numerical results that are accurate to four significant figures when $n>20$ (where Riordan's formula would require excessively heavy computation). *H. S. M. Coxeter* (Toronto, Ont.).

Brătilă, F., and Sergescu, P. Generalized combinations. Revista Mat. Timişoara 21, no. 2, 16 pp. (1941). (Romanian) [MF 15604]

In connection with some work on mathematical biology E. Galvez Laguarta was led to determine the numerical value of the number $C^p(a_1, a_2, \cdots, a_n)$ of combinations with restricted repetitions of the letters $A_1, A_2, \cdots, A_n$, taken p at a time, where A_ν may be taken at most a_ν times. The authors express this number in terms of the corresponding number $K_n^p=C^p(p, p, \cdots, p)$ of combinations with unrestricted repetitions ($K_n^0=1$, $K_n^p=0$ if $p<0$). Their generating function is

$$(1)\qquad (1+x+x^2+\cdots)^n=\sum_{s=0}^{\infty}K_n^s x^s.$$

For combinations with restricted repetitions we have, setting $M=a_1+a_2+\cdots+a_n$, the generating function

$$\sum_{p=0}^{M}C^p(a_1, \cdots, a_n)x^p=\prod_i(1+x+\cdots+x^{a_i})$$
$$=\prod_i\{(1-x^{a_i+1})(1+x+x^2+\cdots)\}=(1+x+\cdots)^n$$
$$-\sum x^{a_1+1}(1+x+\cdots)^n+\sum x^{a_1+a_2+2}(1+x+\cdots)^n$$
$$-\cdots\pm x^{a_1+\cdots+a_n+n}(1+x+\cdots)^n.$$

In view of (1) we get, on comparing coefficients of x^p,

$$C^p(a_1, \cdots, a_n)=K_n^p-\sum K_n^{p-a_1-1}+\sum K_n^{p-a_1-a_2-2}$$
$$-\cdots\pm K^{p-a_1-\cdots-a_n-n}.$$

Since $C^p(a_1, \cdots, a_n)=C^{M-p}(a_1, \cdots, a_n)$, we may also write

$$C^p(a_1, \cdots, a_n)=K_n^{M-p}-\sum K_n^{M-p-a_1-1}$$
$$+\sum K_n^{M-p-a_1-a_2-2}-\cdots\pm K^{M-p-a_1-\cdots-a_n-n}.$$

This identity is the chief result of the paper. The authors derive it by direct combinatorial methods using complete induction on n. Since $K_n^s=\binom{n+s-1}{s}$, an explicit determination of $C^p(a_1, \cdots, a_n)$ in terms of binomial coefficients is obtained. *I. J. Schoenberg* (Philadelphia, Pa.).

Walther, A. Zum Determinantenverfahren von Chiò. Z. Angew. Math. Mech. 24, 41 (1944). [MF 15848]

The author establishes the equivalence of a method of reducing an nth order determinant to an $(n-1)$th order determinant [Chiò's method; cf. Whittaker and Robinson, The Calculus of Observations, 3d ed., Blackie, London-Glasgow, 1940] and a method of Gauss for reducing a system of n nonhomogeneous linear equations in n unknowns to a system of $n-1$ equations in $n-1$ unknowns.

A. J. Kempner (Boulder, Colo.).

Hua, Loo-Keng. Orthogonal classification of Hermitian matrices. Trans. Amer. Math. Soc. 59, 508–523 (1946). [MF 16469]

Two Hermitian matrices H and H_1 are conjunctive orthogonally if there exists an orthogonal matrix P such that $PHP'=H_1$. It follows immediately that, if H and H_1 are conjunctive orthogonally, the two matrices $H\bar{H}$ and $H_1\bar{H}_1$ are similar so that the elementary divisors of the two matrices are the same. The author proves (i) that each elementary divisor corresponding to a negative characteristic root of $H\bar{H}$ must occur an even number of times and that complex elementary divisors must occur in conjugate pairs and (ii) that the elementary divisors of $H\bar{H}$, together with a sign attached to each positive elementary divisor, determine H apart from a conjunctive orthogonal transformation. In doing this he considers the following more general problem. Let H and H_1 be two Hermitian matrices and S and S_1 two nonsingular symmetric matrices. If there exists a nonsingular matrix P such that $\bar{P}HP'=H_1$ and $PSP'=S_1$, the two pairs H, S and H_1, S_1 are equivalent. (If $S=S_1=I$, the identity matrix, H and H_1 are conjunctive orthogonally.) The author finds a canonical form for the pair H_1, S_1 and shows that it is not determined by the elementary divisors alone.

He concludes by considering the group of automorphs of (H, S), that is, the group of nonsingular matrices P satisfying $\bar{P}HP'=H$ and $PSP'=S$, and proves that, if the characteristic roots of $HS^{-1}\bar{H}\bar{S}^{-1}$ are α_i with multiplicities p_i, $\alpha_i>0$; $\beta_j, \bar{\beta}_j$ with multiplicities q_j, q_j, β_j complex; and γ_k with multiplicities $2r_k$, $\gamma_k<0$, then the group of automorphs of (H, S) depends on at least $\sum r_k$ parameters.

J. Williamson (Flushing, N. Y.).

Dmitriev, N., and Dynkin, E. On the characteristic numbers of a stochastic matrix. C. R. (Doklady) Acad. Sci. URSS (N.S.) 49, 159–162 (1945). [MF 16403]

A matrix (p_{ij}) of order n is a stochastic matrix if every $p_{ij}\geqq 0$ and $\sum_j p_{ij}=1$ $(i=1, 2, \cdots, n)$. The authors study the domain of characteristic roots of such matrices and give simple geometrical proofs of a number of theorems, most of which have already been established by Romanovsky [Acta Math. 66, 147–251 (1936)]. In particular, they prove that $\lambda=1$ is a root of any stochastic matrix P, that all other roots are in the circle $|\lambda|\leqq 1$, and that a root λ with $|\lambda|=1$ is of the form $e^{2\pi i p/q}$, where $q\leqq n$. *N. H. McCoy.*

Comessatti, Annibale. Sulla normalizzazione delle forme bilineari alternate a coefficienti interi. Boll. Un. Mat. Ital. (2) 5, 61–72 (1943). [MF 16088]

The author reduces an alternating bilinear form to the normal form of Frobenius. *C. C. MacDuffee.*

Zin, Giovanni. Sui sistemi di equazioni lineari a coefficienti dipendenti linearmente da un parametro. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. 79, 209–219 (1944). [MF 16234]

Let $\lambda_1, \lambda_2, \cdots, \lambda_l$ be the distinct roots of $|A+\lambda B|=0$ and let $\|A+\lambda_i B\|$ be of nullity μ_i. Select in any way μ_i linearly independent solution vectors of $(A+\lambda_i B)X=0$. Then all of these $\mu_1+\cdots+\mu_l$ vectors are linearly independent. *C. C. MacDuffee* (Madison, Wis.).

Bell, Clifford. A note on one-dimensional linear transformations. Tôhoku Math. J. 48, 55–59 (1941). [MF 16346]

The author finds three involutions T_1, T_2, T_3, of which the first is $x'=(ax+b)/(cx-a)$, such that $T_iT_j=T_k$ for i, j, k distinct. *C. C. MacDuffee* (Madison, Wis.).

Villa, Mario. Il gruppo delle trasformazioni pseudoproiettive. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. 77, 242–250 (1942). [MF 16251]

The author discusses pseudoprojectivities in S_r (with particular reference to the cases $r=1, 2$), that is, point-transformations of the type

$$y_j=\frac{\sum_i a_{ji}x_i+\sum_i b_{ji}\bar{x}_i+c_i}{\sum_i k_i x_i+\sum_i \bar{k}_i\bar{x}_i+h}, \qquad i, j=1, \cdots, r,$$

where h is real, a_{ji}, b_{ji}, c_i, k_i may be complex, and the bar denotes the conjugate complex. He remarks that the Lorentz transformation can be interpreted as a pseudoprojectivity in S_2 and that all such transformations leave invariant a hyperalgebraic variety with equations $2X\bar{X}+Y^2+\bar{Y}^2=0$. Here $X=y+iz$, $Y=x+ict$, where x, y, z, t are coordinates in space-time and c is the velocity of light. *J. A. Todd.*

Milgram, A. N. Saturated polynomials. Rep. Math. Colloquium (2) 7, 65–67 (1946). [MF 16136]

A polynomial with coefficients (mod p) is called saturated if it is the least common multiple of $(x^{r_1}-x)^{m_1}, \cdots, (x^{r_k}-x)^{m_k}$, where $r_i=p^{n_i}$, p is a prime, and k, n_i, m_i are arbitrary positive integers. Answering a question raised by Menger, the author proves that, if $f(x)$ and $\phi(x)$ are arbitrary polynomials with coefficients (mod p), then $f(\phi(x))=0$ (mod $f(x)$) if and only if $f(x)$ is saturated. It is remarked that with a slight modification the proof is valid in the case that the coefficients belong to any $GF(p^t)$; in the general statement p is replaced by p^t. *L. Carlitz* (Durham, N. C.).

Abstract Algebra

Shoda, Kenjiro. Über die allgemeinen algebraischen Systeme. I. Proc. Imp. Acad. Tokyo 17, 323–327 (1941). [MF 14714]

"The object of this paper is a systematic introduction of the concepts of modern algebra, lattices, groups, rings, free groups, free products, etc." An algebraic system is defined by the existence of certain combination operations $a\alpha b$ which give other elements in the set. Isomorphism, homomorphism and residue systems are introduced generally. A system is primitive when the operations can be performed without exceptions and the homomorphic systems satisfy the same given set of relations. Free algebraic systems are introduced and a theorem analogous to a theorem by Tietze on relations in free groups is derived. General concepts of embedding and extension of systems as well as free products are discussed. *O. Ore* (New Haven, Conn.).

Shoda, Kenjiro. Über die allgemeinen algebraischen Systeme. II. Proc. Imp. Acad. Tokyo 18, 179–184 (1942). [MF 14752]

[Cf. the preceding review.] This paper contains general remarks of various kinds regarding free groups, lattices and definitions of various noncommutative polynomials with various commuting relations. Operators in general systems are also introduced. *O. Ore* (New Haven, Conn.).

Shoda, Kenjiro. Über die allgemeinen algebraischen Systeme. III. Proc. Imp. Acad. Tokyo 18, 227–232 (1942). [MF 14757]

[Cf. the preceding review.] In this part one finds the laws of homomorphism and isomorphism for general algebraic systems as well as the analogues of the Jordan-Hölder theorem and Schreier's refinement theorem for groups. *O. Ore* (New Haven, Conn.).

Shoda, Kenjiro. Über die allgemeinen algebraischen Systeme. IV. Proc. Imp. Acad. Tokyo 18, 276–279 (1942). [MF 14761]

[Cf. the preceding review.] In this part one finds a discussion of the analogue of direct products for general systems. An analogue of the Schmidt-Remak theorem on groups is deduced from a result given by the reviewer. The relation between direct product and direct union is investigated. Simple and completely irreducible systems are introduced. *O. Ore* (New Haven, Conn.).

Shoda, Kenjiro. Über die allgemeinen algebraischen Systeme. V. Proc. Imp. Acad. Tokyo 19, 114–118 (1943). [MF 14800]

[Cf. the preceding review.] This part contains the analogue of certain theorems on solvable groups. When $\mathfrak{A}$ is an algebraic system one can identify certain elements according to some family B of new relations. One obtains a system $\mathfrak{A}_B$ which is isomorphic to some quotient system $\mathfrak{A}/\mathfrak{A}'$. (Example. By introducing commutativity relations in a group, $\mathfrak{A}'$ becomes the commutator group.) By identifying the elements in $\mathfrak{A}'$ according to B one obtains $\mathfrak{A}'_B\cong\mathfrak{A}'/\mathfrak{A}''$. When this process leads to the zero element the system $\mathfrak{A}$ is said to be B-solvable. Various analogues to the theorems on solvable groups are derived. One can introduce central series and nilpotent systems analogous to nilpotent groups. *O. Ore* (New Haven, Conn.).

Shoda, Kenjiro. Über die allgemeinen algebraischen Systeme. VI. Proc. Imp. Acad. Tokyo 19, 259–263 (1943). [MF 14819]

Part VI contains a continued discussion of solvable systems as defined and introduced in part V [cf. the preceding review]. *O. Ore* (New Haven, Conn.).

Shoda, Kenjiro. Über die allgemeinen algebraischen Systeme. VII. Proc. Imp. Acad. Tokyo 19, 515–517 (1943). [MF 14847]

[Cf. the preceding review.] Here the author introduces the concept of a lattice quasi-graduated with respect to union. In such lattices there exists a number $d(a, b)\geqq 1$

associated with each quotient a/b such that $d(a/a)$ is finite and the following conditions hold. (1) When $a \geqq b \geqq c$,

$$d(a/b) \leqq d(a, c), \quad d(b/c) \leqq d(a, c), \quad d(a/c) \leqq d(a/b) \cdot d(b/c).$$

(2) When $a \geqq b$, $a' \geqq b$, $d(a \cup a'/b) \leqq d(a/b) \cdot d(a'/b)$. The lattice is graduated when $a \geqq b > c$ implies $d(a/b) < d(a/c)$ and $a > b \geqq c$ implies $d(b/c) < d(a/c)$.

If $a \geqq b_i$ for every b_i in the set $L = \{b_i\}$ and $b = b_1 \cup b_2 \cup \cdots$ has the property that $d(a/b)$ is finite the element a is said to be algebraic over L. This is applied to algebraic systems in which it is supposed that the lattice of subsystems is quasi-graduated. If $d(L/L')$ is finite for two subsystems $L > L'$ then L is said to be a finite extension of L'. An element a is algebraic over L when it lies in some finite extension of L.

An element a is dependent on the subset E when there exists a relation between a and a finite subset F of E which is not equivalent to (derivable from) a relation in F. A subset E is independent when every element a is independent of $E-a$. Simple facts regarding these concepts are derived. In the case of linear dependence only the elements in the system generated by the elements of E shall be algebraic over E. *O. Ore* (New Haven, Conn.).

Shoda, Kenjiro. Über die allgemeinen algebraischen Systeme. VIII. Proc. Imp. Acad. Tokyo **20**, 584–588 (1944). [MF 14929]

[Cf. the preceding review.] In this part there is a further discussion of systems in which linear dependence is defined. Under certain limiting conditions it follows that such systems are completely reducible and the direct product of systems generated by a single element. An algebraic operator system is also definable. To conclude, there are remarks about algebraic systems in which the operations may not be performed for all pairs of elements and also general statements about representations and representation systems. *O. Ore* (New Haven, Conn.).

Everett, C. J., and Ulam, S. Projective algebra. I. Amer. J. Math. **68**, 77–88 (1946). [MF 15491]

The authors seek to characterize abstractly those properties of the Boolean algebra of all subsets of a direct product $X \times Y$ which are connected with the existence, in such an algebra, of the following operations: (1) projection on both factors X, Y (identified with subsets of $X \times Y$); (2) direct product of subsets of X and Y. More generally, they call a subalgebra of the Boolean algebra of all subsets of $X \times Y$, which is closed under those operations, a projective algebra of subsets of $X \times Y$. They first set up a system of postulates for projective algebras, from which they derive abstractly the familiar properties of projection and direct product. In the second place, they show that any projective algebra may be embedded in another one which is complete (in the sense of unrestricted union and intersection). Finally they prove that any abstract projective algebra which is atomic is isomorphic (with preservation of the operations of projection and direct product) to a subalgebra of the projective algebra of all subsets of a suitably chosen direct product $X \times Y$. *J. Dieudonné.*

Schützenberger, Marcel-Paul. Sur les structures de Dedekind. C. R. Acad. Sci. Paris **218**, 818–819 (1944). [MF 15328]

In a previous paper [same C. R. **216**, 717–718 (1943); these Rev. **5**, 226] the author has introduced a lower generating element in a structure as an element having a single predecessor, that is, an element which cannot be represented properly as a union. Upper generating elements are defined analogously. In the present paper the author announces the theorem that in any finite Dedekind structure the number of generating elements of the two kinds is the same. *O. Ore* (New Haven, Conn.).

Kobayasi, Masatada. On the axioms of the theory of lattice. Proc. Imp. Acad. Tokyo **19**, 6–9 (1943). [MF 14791]

In place of the usual absorptive law of lattice theory, the weaker L4: (a) if $y \cup x = x$, then $y \cap x = y$; (b) if $y \cap x = x$, then $y \cup x = y$ is introduced. This, together with the commutativity and associativity of $\cup$ and $\cap$, and the idempotency of $\cup$: L1: $x \cup x = x$; L2: (a) $x \cup y = y \cup x$; (b) $x \cap y = y \cap x$; L3: (a) $x \cup (y \cup z) = (x \cup y) \cup z$; (b) $x \cap (y \cap z) = (x \cap y) \cap z$ is shown to form an independent set of postulates for the concept of lattice. *A. L. Foster.*

Yosida, Kôsaku, and Fukamiya, Masanori. On vector lattice with a unit. II. Proc. Imp. Acad. Tokyo **17**, 479–482 (1941). [MF 14731]

[For part I, by Yosida, see the same Proc. **17**, 121–124 (1941); these Rev. **3**, 210.] The theory of a real vector lattice A with "unit" (an element u such that every x in A satisfies $-nu < x < nu$ for some positive integer n) is found to have many analogies with the theory of rings [see also M. H. Stone, Proc. Nat. Acad. Sci. U. S. A. **27**, 83–87 (1941); these Rev. **2**, 318]. Like the nilpotent elements in ring-theory, the infinitesimals (the elements x such that $nx < u$ for all integers n) are precisely the elements common to all maximal ideals (in a vector lattice A, the latter correspond biunivocally to the homomorphic mappings of A into the reals) and thus constitute what may be called the radical of A. An example which lacks a unit shows that in general "infinitesimal" elements are not so simply characterized. *M. H. Stone* (Chicago, Ill.).

Yosida, Kôsaku. On the representation of the vector lattice. Proc. Imp. Acad. Tokyo **18**, 339–342 (1942). [MF 14765]

An arbitrary real vector-lattice A is discussed with the help of a fixed maximal set of mutually disjoint elements u_α (the elements x and y are disjoint if $|x| \cap |y| = 0$). The class of all x such that $n(|x| \cap u_\alpha) < u_\alpha$ for all integers n and all u_α is termed the radical of A. The ideals in A corresponding to homomorphic mappings of A into simply-ordered real vector spaces are called prime. Each prime ideal N singles out an element $u_\alpha = u_N$ not mapped into 0 by the corresponding homomorphism and associates with each x an extended-real function of N defined by the equation $x(N) = \sup(\lambda; \lambda u_N \leqq x)$, the infinite values $+\infty$ and $-\infty$ being admitted. Apart from the ambiguities occasioned by the infinite values, the correspondence from x to $x(N)$ is a homomorphic mapping of A into the extended real number system and determines a representation of A by the functions of N associated with its various elements x. The equation $x(N) = 0$ identically in N characterizes the radical. The set of all N can be so topologized that the functions $x(N)$ become continuous in N wherever they are finite. If A is Archimedean in the sense that $ny \leqq x$ for $x \geqq 0$ and all positive n implies $y \leqq 0$, then the radical consists of 0 alone and the set where $x(N) = \pm\infty$ is nowhere dense in the topology mentioned above. [See also M. H. Stone, Proc. Nat. Acad. Sci. U. S. A. **27**, 83–87 (1941); these Rev. **2**, 318.] *M. H. Stone* (Chicago, Ill.).

Cohen, I. S., and Seidenberg, A. Prime ideals and integral dependence. Bull. Amer. Math. Soc. **52**, 252–261 (1946). [MF 16187]

Let $\mathfrak{S}$ be a commutative ring containing the ring $\mathfrak{R}$ and having the same identity element as $\mathfrak{R}$. The prime ideal $\mathfrak{P}$ in $\mathfrak{S}$ lies over the prime ideal $\mathfrak{p}$ in $\mathfrak{R}$ if $\mathfrak{P}\cap\mathfrak{R}=\mathfrak{p}$. If for every prime ideal $\mathfrak{p}$ in $\mathfrak{R}$ there exists a prime ideal $\mathfrak{P}$ in $\mathfrak{S}$ lying over $\mathfrak{p}$, one may say that the "lying over" theorem holds for the rings $\mathfrak{R}$ and $\mathfrak{S}$. Let $\mathfrak{p}$ and $\mathfrak{q}$ be arbitrary prime ideals in $\mathfrak{R}$ such that $\mathfrak{q}\subset\mathfrak{p}$. The "going up" theorem may be said to hold for the rings $\mathfrak{R}$ and $\mathfrak{S}$ if for every prime ideal $\mathfrak{Q}$ in $\mathfrak{S}$ lying over $\mathfrak{q}$ there exists a prime ideal $\mathfrak{P}$ lying over $\mathfrak{p}$ and such that $\mathfrak{Q}\subset\mathfrak{P}$. If, for every prime ideal $\mathfrak{P}$ in $\mathfrak{S}$ lying over $\mathfrak{p}$, there exists a prime ideal $\mathfrak{Q}$ in $\mathfrak{S}$ lying over $\mathfrak{q}$ such that $\mathfrak{Q}\subset\mathfrak{P}$, then the "going down" theorem holds. The authors prove the "lying over" and "going up" theorems under the assumption that $\mathfrak{S}$ is integrally dependent on $\mathfrak{R}$. The "going down" theorem is also established under certain additional assumptions. These theorems are extensions of results due to Krull [Math. Z. **42**, 745–766 (1937)] for rings without divisors of zero. Examples are given to show that some of the results are the best possible.

N. H. McCoy (Northampton, Mass.).

Loonstra, F. A theorem about ideals in commutative rings. Nederl. Akad. Wetensch., Proc. **49**, 39–40 = Indagationes Math. **8**, 3–4 (1946). [MF 16560]

It is observed, in effect, that an integral domain with basis condition must be a principal ideal ring if every pair of its elements generates a principal ideal. It is noted also that the basis assumption cannot be omitted.

I. S. Cohen (Philadelphia, Pa.).

Abe, Makoto, und Nakayama, Tadasi. Über die Irreduzibilität und absolute Irreduzibilität des Darstellungsmoduls. Proc. Imp. Acad. Tokyo **20**, 274–277 (1944). [MF 14893]

Let $\mathfrak{m}$ be a (not necessarily finite dimensional) vector space over a field P. Let $\mathfrak{m}$ admit also a domain $\mathfrak{S}$ of operators which commute with those of P, and suppose that $\mathfrak{m}$ is irreducible as an $(\mathfrak{S}, P)$-module. Let K be a subfield containing P of the quasifield Δ of endomorphisms of $\mathfrak{m}$. Then $\mathfrak{m}$ is absolutely irreducible, considered as an $(\mathfrak{S}, K)$-module, if and only if K is a maximal subfield of Δ. This had previously been proved in the finite-dimensional case by Abe [Proc. Phys.-Math. Soc. Japan (3) **24**, 769–789 (1942); these Rev. **7**, 361].

I. S. Cohen.

Artin, E., and Whaples, G. A note on axiomatic characterization of fields. Bull. Amer. Math. Soc. **52**, 245–247 (1946). [MF 16185]

It is shown that the fields of class field theory can be characterized by axioms weaker than those used earlier by the same authors [same Bull. **51**, 469–492 (1945); these Rev. **7**, 111]. In axiom 1 of their earlier paper it was assumed that $|\alpha|_{\mathfrak{p}}=1$ for all but a finite set of $\mathfrak{p}$. This condition is now replaced by the assumption that for every $\alpha\neq 0$ of k the product of all valuations converges absolutely to the limit 1, and the corresponding axiom is called axiom 1*. Certain adjustments in the definitions and proofs are required because of this modification. It is finally concluded that axiom 1* is a consequence of axioms 1 and 2. An example shows that axiom 1* can be fulfilled without axioms 1 and 2. Two further examples illustrating the significance of axioms 1 and 2 are discussed.

O. Todd-Taussky (London).

THEORY OF GROUPS

✱Speiser, Andreas. Die Theorie der Gruppen von endlicher Ordnung. Dover Publications, New York, N. Y., 1945. x+262 pp. $3.50.

Photographic reproduction of the third edition [Springer, Berlin, 1927].

Piccard, Sophie. Des conditions nécessaires et suffisantes pour qu'un système de substitutions indépendantes engendre un groupe régulier. C. R. Acad. Sci. Paris **222**, 716–718 (1946). [MF 16174]

Let $S_1, S_2, \cdots, S_m$ be a system of permutations of the n letters $1, 2, \cdots, n$ which are independent in the sense that none of them can be expressed in terms of the others. The author states without proof necessary and sufficient conditions that the group generated by these permutations be a regular permutation group. These conditions are quite complicated and are expressed for the most part in terms of "domains of connexion" defined as follows: a nonvoid (but not necessarily proper) subset E of the letters $1, 2, \cdots, n$ is said to be a domain of connexion of $S_{i_1}, S_{i_2}, \cdots, S_{i_k}$, $k\leqq m$, if E contains the letters of at least one cycle of each of $S_{i_1}, \cdots, S_{i_k}$, while no proper subset of E has this property. Typical of the eight conditions imposed are: (a) if the permutations $S_{i_1}, \cdots, S_{i_k}$, $k<m$, generate a subgroup of order μ, then each of the various domains of connexion of $S_{i_1}, \cdots, S_{i_k}$ contains exactly μ letters; (b) if $S_1, \cdots, S_{i-1}, S_{i+1}, \cdots, S_m$ generate a subgroup G_i, and if the number of different domains of connexion of $S_1, \cdots, S_{i-1}, S_{i+1}, \cdots, S_m$ is l, then $S_i^f \epsilon G_i$ and $S_i^f S_j S_i^{-f} \epsilon G_i$ for $j\neq i$ and $f=1, 2, \cdots, l-1$.

S. A. Jennings (Vancouver, B. C.).

Shoda, Kenjiro. Über die Schreiersche Erweiterungstheorie. Proc. Imp. Acad. Tokyo **19**, 518–519 (1943). [MF 14848]

Let N and B be two groups. A group A is an extension of N by B when A contains N as a normal subgroup such that the factor group A/N is isomorphic to B. The author constructs such extensions when B is introduced as a free group of a certain number of generators with a given set of relations.

O. Ore (New Haven, Conn.).

no 8.

Kořínek, Vladimír. Bemerkung über charakteristisch einfache Gruppen. Věstník Královské České Společnosti Nauk. Třída Matemat.-Přírodověd. **1940**, 8 pp. (1940). (German. Czech and English summaries) [MF 16123]

In this paper the author proves the following results. Let G be a group which has a decomposition into a direct product $(*)$ $G=\prod G_\sigma$, $0\leqq\sigma<\varphi$, φ a fixed cardinal, satisfying the following conditions. (1) Two arbitrary direct factors G_α and G_β of $(*)$ always possess direct factors H_α of G_α and H_β of G_β which are different from the identity group and which are isomorphic with each other. (2) All the direct factors of the decomposition $(*)$ are characteristically simple. Then the group G is itself characteristically simple. Conversely, let G be a characteristically simple group; then every decomposition of G into a direct product satisfies the

condition (1). Whether the condition (2) is also necessary for a characteristically simple group is left open.

H. F. Tuan (Peiping).

Birkhoff, Garrett. On groups of automorphisms. Revista Unión Mat. Argentina 11, 155–157 (1946). (Spanish) [MF 16370]

The author demonstrates very simply three converses to the well-known theorem that the automorphisms of each abstract algebra form a group. Indeed, each abstract group G with α elements is isomorphic to the group of all isomorphisms of each of the following: (1) an abstract algebra G, of α elements and α unitary operations; (2) some partially ordered system X of $\alpha^2+\alpha$ elements; (3) a distributive lattice B^X with at most $2^{\alpha^2+\alpha}$ elements. For the latter two cases the proof uses the axiom of choice.

A. A. Bennett (Providence, R. I.).

Tchernikoff, S. On infinite special groups with finite centers. Rec. Math. [Mat. Sbornik] N.S. 17(59), 105–130 (1945). (Russian. English summary) [MF 14596]

Les groupes spéciaux d'après la définition de l'auteur [même Rec. N.S. 6(48), 199–214 (1939); ces Rev. 1, 162; voir aussi le même Rec. 7(49), 35–64, 539–548 (1940); ces Rev. 2, 5, 126], généralisant la définition d'O. Schmidt, sont les groupes qui remplissent les conditions suivantes: (1) chaque chaîne décroissante de sous-groupes est fini; (2) ils possèdent des séries centrales croissantes.

Dans le mémoire présent l'auteur s'occupe des p-groupes spéciaux et, plus généralement, des groupes qui sont p-extensions finies (c'est-à-dire, des extensions au moyen de p-groupes-facteurs finis) de p-groupes abéliens arbitraires. Au cas des groupes à centre fini ces deux types coïncident. L'auteur réussit à en donner des exemples, c'est-à-dire, à construire des p-groupes spéciales infinis à centre fini. Il étudie ses propriétés et donne des conditions nécessaires et suffisantes pour que les types mentionnés aient leur centre fini. Enfin il construit des p-groupes infinis qui possèdent des séries centrales croissantes, mais qui ne possèdent pas des séries centrales décroissantes.

H. Freudenthal.

Courtois, Jacques. Les expanseurs. C. R. Acad. Sci. Paris 222, 377–378 (1946). [MF 16017]

P. A. M. Dirac has defined "expansors" and has thus obtained a representation of the Lorentz group by unitary matrices of infinitely many dimensions [Proc. Roy. Soc. London. Ser. A. 183, 284–295 (1945); these Rev. 6, 145]. Here the method and result are generalized so that the author obtains representations of the affine group in a space of arbitrary dimension, finite or infinite.

A. Schwartz.

Courtois, Jacques. Réductibilité des expanseurs. C. R. Acad. Sci. Paris 222, 480–482 (1946). [MF 16028]

By making use of a metric tensor, the author reduces an expansor to an infinite set of tensors in certain special cases. To avoid convergence difficulties one is restricted to spaces with positive definite fundamental forms and expansors of order $u=2p/q$, p and q integers. The method is applicable to a spinor-expansor only if its order is an integer.

A. Schwartz (State College, Pa.).

Turri, Tullio. Sopra sostituzioni lineari cicliche nella teoria dei gruppi continui finiti semplici. Rend. Sem. Fac. Sci. Univ. Cagliari 13, 59–74 (1943). [MF 16223]

Let G be a semi-simple continuous group of order r and rank l. The characteristic roots of G may be obtained from those relative to an Abelian subgroup of G of order l by means of a linear substitution Σ [cf. É. Cartan, Sur la Structure des Groupes de Transformations Finis et Continus, Paris thesis, 1894; 2d ed., Vuibert, Paris, 1933; pp. 58 ff.]. The author obtains Σ explicitly for each of the nine types of simple groups, calculates the characteristic equation and index μ of Σ and thus verifies that for these groups the relation $l\mu=r-l$ holds.

S. A. Jennings.

de Groot, J. Space groups and their axioms. Nederl. Akad. Wetensch., Proc. 49, 156–161 = Indagationes Math. 8, 53–58 (1946). [MF 16568]

The definition of space-groups in terms of neighborhoods is obtained from that for topological groups, except that no finite intersection or separation property is assumed for the neighborhoods. It is found that the implications among separation axioms are more or less as for topological groups.

R. Arens (Princeton, N. J.).

Weil, André. Sur quelques résultats de Siegel. Summa Brasil. Math. 1, 21–39 (1946). [MF 16542]

This paper exploits homogeneous spaces to simplify some results of Siegel. The first (and longer) part of the paper proves a formula of Siegel [Ann. of Math. (2) 46, 340–347 (1945); these Rev. 6, 257] concerning the group G of $n\times n$ matrices (real elements) of determinant 1, the subgroup Γ of those with integer elements, and the set of lattices in Euclidean n-space E^n. A lattice R in E^n is any transform by an element of G of the set of points in E^n all of whose coordinates are integers. Siegel's formula asserts, for $f(x)$ (Riemann integrable) defined on E^n, that $\int f(x)dx=\int\sum f(x)dR$, where the summation is over all $x\neq 0$ in R, so that this sum is a function of R.

The author first proves a general integration formula involving groups $G\supseteq g\supseteq\gamma$, where g,γ are closed in the locally compact group G and all three are unimodular (left and right Haar measures coincide). It asserts that the integral over G/γ equals the iterated integral over g/γ and G/g, the integrals involved being those defined on a general homogeneous space by Weil [L'intégration dans les groupes topologiques et ses applications, Actual. Sci. Ind., no. 869, Hermann, Paris, 1940; these Rev. 3, 198]. Then he considers still another subgroup Γ (closed and unimodular) with $G\supseteq\Gamma\supseteq\gamma$ and as a special case of the above obtains (in case g/γ has finite measure) for $f(x)$ defined on G/g

$$\int_{G/g} f(x)dx=c\int_{G/\Gamma} d\bar{\xi}\int_{\Gamma/\gamma} f(x\xi)d\xi,$$

where c is a constant. This is the general formula which yields Siegel's, by taking G,Γ as in the first paragraph above, choosing g properly so that $G/g=E^n$, and defining $\gamma=\Gamma\cap g$. To obtain Siegel's formula it remains to (1) show that g/γ has finite measure (easy), (2) turn the integral over Γ/γ into $\sum f(x)$ (done following Siegel) and (3) eliminate the constant in Weil's formula (difficult). This last includes proving that G/Γ has finite measure.

The second part of the paper improves other theorems of Siegel [Ann. of Math. (2) 44, 674–689 (1943); these Rev. 5, 228]. A continuous mapping is called proper if the inverse image of a compact set is compact. For a locally compact group G with closed subgroups Γ and g the natural representation of Γ on G/g is called proper if each transformation of the representation is proper. It is proved that, for G, Γ unimodular, and G/Γ of finite measure, Γ is proper on G/g if and only if g is compact.

W. Ambrose.

Braconnier, Jean. Sur les modules localement compacts. C. R. Acad. Sci. Paris **222**, 527–529 (1946). [MF 16031]

Let (A, M) be a topological left A-module, that is, M is a topological Abelian group, A is a topological ring, and am $(a\epsilon A, m\epsilon M)$ is a continuous function; similarly let (M, A) denote a topological right A-module. As a module under left or right multiplication, A is denoted by A_L or A_R. Now take A and M locally compact, and let M^* be the dual group of M. A right module (M^*, A) is defined by $m^*a=(am)^*$, and denoted by $(A, M)^*$; similarly from an (M, A) is defined an $(A, M^*)=(M, A)^*$. Statements made include: (1) $(A, M)^{**}=(A, M)$, (2) if k is the component of 0 in A and f is the union of all compact subgroups in A then $k/(k\cap f)$ is an algebra of finite rank over the reals, (3) if A is a nondiscrete sfield (noncommutative field) then $(A_L)^*=A_R$. *W. Ambrose* (Ann Arbor, Mich.).

Markoff, A. On unconditionally closed sets. Rec. Math. [Mat. Sbornik] N.S. **18(60)**, 3–28 (1946). (Russian. English summary) [MF 16674]

The following extracts from the author's summary give the substance of the paper.

This paper is devoted to the following problem: to prove or to refute the proposition that every unconditionally closed subset of a group is algebraic. The notions of an unconditionally closed set and of an algebraic set here involved are defined as follows. (1) Let G be a group, $A\subset G$. We say that A is unconditionally closed in G if A is closed in every topology of G. (2) Let m be a positive integer; let Φ be a function of m elements of the group G with values in G. We say that Φ is multiplicative if this function can be represented in the form

$$\Phi(x_1, \cdots, x_m)=\prod_{i=1}^{n} x_{j_i}^{\epsilon_i},$$

where $j_i=1, \cdots, m$; $\epsilon_i=\pm 1$ and n is a nonnegative integer. (3) Let G be a group, $A\subset G$. We say that A is an elementary algebraic subset of G if there exists a multiplicative function Φ of $m+1$ arguments $(m\geqq 0)$ such that

$$A=\mathcal{E}(\Phi(a_1, \cdots, a_m, x)=1_G),$$

where $a_1, \cdots, a_m$ are suitably chosen elements of G and 1_G denotes the unit element of G. (4) We say that A is an additively algebraic subset of G if A is the set-theoretical sum of a finite number of elementary algebraic subsets of G. (5) We say that A is an algebraic subset of G if A is the intersection of a nonempty family of additively algebraic subsets of G.

The following theorem can be easily proved. Theorem 1. Every algebraic subset of the group G is unconditionally closed. The author suspects that the inverse is also true. We deduce from theorem 1 the following corollary 1. The closure of a subset A of the group G in any topology of this group is contained in the algebraic closure of A in G. We prove the following theorem which completes this result in the domain of countable groups. Theorem 2. Whatever be the subset A of the countable group G, there exists a topology of this group such that the closure of A in this topology coincides with the algebraic closure of A in G. This theorem implies the required inversion of theorem 1 for countable groups. We obtain corollary 2. Every set unconditionally closed in the countable group G is algebraic in this group. The proof of theorem 2 is carried out by constructing an appropriate multinorm in the group G [cf. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **9**, 3–64 (1945); these Rev. **7**, 7]. *L. Zippin.*

✱Chevalley, Claude. Theory of Lie Groups. I. Princeton Mathematical Series, vol. 8. Princeton University Press, Princeton, N. J., 1946. ix+217 pp. $3.00.

This book forms a companion to the well-known volumes by Weyl [The Classical Groups. Their Invariants and Representations, 1939; these Rev. 1, 42] and Pontrjagin [Topological Groups, 1939; these Rev. 1, 44] in the same series. The author first defines the linear, orthogonal, unitary and symplectic groups and discusses their topology. Thus he shows that a regular matrix can be expressed uniquely as the product of a unitary and a positive definite Hermitian matrix. The dimensions of the group spaces, and the infinitesimal matrices generating these groups, are also determined.

Chapter II is on topological groups; most of the results are also in Pontrjagin. A novel feature of Chevalley's treatment is the construction, by a method ascribed to H. Cartan, of a covering space without using curves. Homogeneous spaces and factor groups are correlated rigorously with closed subgroups and (closed) normal subgroups. There is an interesting discussion of Clifford numbers and spinors, attributed to V. Bargmann.

An original discussion of the geometrical concept of an "analytic manifold" is then given, which permits very explicit, nonintuitive definitions of analytic submanifolds, "tangent spaces," infinitesimal transformations and Lie algebras of infinitesimal transformations; the discussion is almost entirely freed from the countability axiom.

Abstract Lie groups are then defined, and the correspondence between Lie groups and Lie algebras established. A feature of the definition of submanifold is that it permits every Lie subalgebra to correspond to a submanifold. Canonical parameters are then used to prove the known fact that topologically isomorphic real Lie groups are analytically isomorphic. It is proved that the commutator subgroup of any simply connected Lie group is closed. In terms of "analytic manifolds," É. Cartan's calculus of exterior differential forms, the equations of Maurer-Cartan, and group measure are established.

The most original part deals with the representations of a compact group G by complex matrices. Standard arguments show that such representations are completely reducible and that inequivalent irreducible representations have orthogonal characters. The author defines the "representative ring" $R(G)$ of G as the ring of polynomials in the coefficients $a_{ij}(g)$ $(g\epsilon G)$ of representations of G. He shows that $R(G)$ is generated by the coefficients of any isomorphic representation and their complex conjugates. The ring-homomorphisms of $R(G)$ onto the complex field C constitute the "algebraic variety" $M(G)$ of G. This $M(G)$ has a "natural" one-one mapping onto the set of all representations ζ of the set of all representations of G; $M(G)$ is a group under a multiplication defined through this correspondence. If G is the unitary group, $M(G)$ is the full linear group. In general, let G be an n-parameter group; let the Lie algebra of G have a basis $x_1, \cdots, x_n$, with $[x_ix_j]=c_k^{ij}x_k$. Then the Lie algebra of $M(G)$ has a basis $x_1, \cdots, x_n$, $y_1, \cdots, y_n$, with

$$[x_ix_j]=c_k^{ij}x_k, \quad [y_iy_j]=-c_k^{ij}x_k, \quad [x_iy_j]=c_k^{ij}y_k.$$

Topologically, $M(G)$ is the direct product of G and Euclidean n-space. The ζ permutable with the operation of passing to the complex conjugate form a subgroup isomorphic with G (Tannaka's duality theorem). Finally, the existence of isomorphic representations and the fact that $R(G)$ is dense in

the space of continuous functions on G are deduced by the classical procedure of Peter-Weyl.

Lie groups of transformations are not discussed; neither are differentiability conditions on the function of composition, sufficient to define Lie groups. Unfortunately almost no bibliography is given. *G. Birkhoff.*

Bochner, S. Formal Lie groups. Ann. of Math. (2) **47**, 192–201 (1946). [MF 16329]

Démonstration des trois théorèmes fondamentaux de Lie pour des groupes définis par des séries de puissances purement formales à coefficients pris d'un corps de caractéristique zéro. *H. Freudenthal* (Amsterdam).

NUMBER THEORY

★Vinogradov, I. M. Osnovy Teorii Čisel. [Foundations of the Theory of Numbers]. 4th edition. OGIZ, Moscow-Leningrad, 1944. 142 pp. (Russian)

This little book contains a remarkable collection of problems, including many giving estimates of sums of importance in analytic theory of numbers. Many of the problems which appeared in earlier editions are somewhat generalized, and unlike earlier editions, proofs and solutions of all problems are given in a section at the end of the book. The theory itself is standard for a course in elementary number theory, but the exposition is clear and simple. Chapter headings: Theory of divisibility, Number-theoretic functions, Congruences, Congruences of the first degree, Principal theorems on congruences, Quadratic congruences, Primitive roots and indices, Solutions to the problems, Answers to the examples, Table of indices. *G. Pall* (Chicago, Ill.).

Uhler, Horace S. A new result concerning a Mersenne number. Mathematical Tables and Other Aids to Computation **2**, 94 (1946). [MF 16140]

The author announces the result of applying Lucas's test for the primality of Mersenne numbers to the case of $M_{229}=2^{229}-1$. In this case it was found that the 228th term of the sequence $S_1=4$, $S_2=14$, $S_3=194$, $\cdots$, $S_{n+1}=S_n^2-2$, is not divisible by M_{229}. Hence M_{229} is composite. [Incidentally, this result is confirmed by the recent discovery by electronic methods that M_{229} is divisible by 1504073.] *D. H. Lehmer* (Berkeley, Calif.).

Sagastume Berra, Alberto E. New considerations on a Diophantine problem. Math. Notae **5**, 215–224 (1945). (Spanish) [MF 16373]

The problem discussed for radix 10 by Levi and Santaló [Math. Notae **5**, 108–119, 162–171 (1945); these Rev. **7**, 242] is generalized to arbitrary radix. *I. Kaplansky.*

Erdös, Paul, and Niven, Ivan. Some properties of partial sums of the harmonic series. Bull. Amer. Math. Soc. **52**, 248–251 (1946). [MF 16186]

It was shown by the reviewer [T. Nagell, Skr. Norske Vid. Akad. Kristiania. I. **1923**, no. 13, 10–15 (1924)] that $\sum_{k=0}^{n}(m+kd)^{-1}$ cannot be an integer. The authors prove the following theorem of a similar nature. There is only a finite number of integers n for which one or more of the elementary symmetric functions of $1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n}$ is an integer. The proof is based on the result of A. E. Ingham [Quart. J. Math., Oxford Ser. **8**, 255–266 (1937)] that there is a prime between x and $x+x^{5/8}$. The authors assert, without proof, that the same result holds for the elementary symmetric functions of $1/m$, $1/(m+1)$, $\cdots$, $1/n$ and of $1/m$, $1/(m+d)$, $1/(m+2d)$, $\cdots$, $1/(m+nd)$. The second result of this paper is the following theorem. No two partial sums of the harmonic series can be equal; that is, it is not possible that $\sum_{k=m}^{n}k^{-1}=\sum_{k=x}^{y}k^{-1}$. The proof depends on the application of the following theorem of I. Schur [cf. Erdös, J. London Math. Soc. **9**, 282–288 (1934)]. When $n>k$, there is in the set $n, n+1, \cdots, n+k-1$ at least one integer containing a prime divisor greater than k. *T. Nagell* (Uppsala).

Schuh, Fred. Can $n-1$ be divisible by $\phi(n)$ when n is composite? Mathematica, Zutphen. B. **12**, 102–107 (1944). (Dutch) [MF 15718]

It is not the purpose of the author to answer this question. The following four known theorems [D. H. Lehmer, Bull. Amer. Math. Soc. **38**, 745–751 (1932)] are proved by elementary methods. (1) If $n-1$ is divisible by $\phi(n)$, n is odd and not divisible by a square greater than 1, and if p is a prime factor of n, n has no factor $px+1$. (2) If n has the factor 3 and $n-1$ is divisible by $\phi(n)$, $(n-1)/\phi(n)\equiv 1 \pmod 3$. (3) If $n=p_1\cdots p_t$, $p_i-1=2^{\lambda_i}n_i$, n_i odd, $n-1$ divisible by $\phi(n)$, μ the minimum value of λ_i, this value appears an even number of times among the λ_i. (4) $(n-1)/\phi(n)$ diminishes if one or more prime factors of n increase. The last three pages give a discussion from which it is inferred that n has at least eleven factors. *N. G. W. H. Beeger.*

Popoviciu, Tiberiu. On indicators. Gaz. Mat., Bucureşti **51**, 306–313 (1946). (Romanian) [MF 16558]

Let ϕ_m denote the mth iterate of the Euler ϕ-function. The author shows that $\phi_m(ab)\geqq\phi_m(a)\phi_m(b)$. *I. Kaplansky* (Princeton, N. J.)

Kössler, M. Einige Sätze aus der elementaren Zahlentheorie. Věstník Královské České Společnosti Nauk. Třída Matemat.-Přírodověd. **1942**, 18 pp. (1942). [MF 16114]

Two identities are set up as follows. (1) Let $q_1<q_2<q_3<\cdots$ denote an arbitrary sequence of positive integers; put $d(n, q_k)=[n/q_k]-[(n-1)/q_k]$, so that $d=1$ or 0 according as n is or is not divisible by q_k. Let $1\leqq n\leqq N$. For arbitrary $f(n)$, $v(n)$ put

$$V(k)=v(k)+v(2k)+m+v(k[n/k]),$$
$$F(n)=\sum_{q_k\mid n}f(q_k)=\sum_{q_k\leqq N}f(q_k)d(n, q_k).$$

Then the following identity holds:

$$\sum_{q_k\leqq N}f(q_k)V(q_k)=\sum_{n=1}^{N}F(n)v(n).$$

(2) Let $f(n)$ and $g(n)$ be two arbitrary functions of n, $f(0)=0$; let $G(r)=\sum_1^r g(k)$, ρ an integer such that $1\leqq\rho<N$, $r=[N/(\rho+1)]$. Then

$$\sum_1^N g(k)f[N/k]=\sum_1^r g(k)f[N/k] +\sum_1^\rho\{f(k)-f(k-1)\}G[N/k]-f(\rho)G(r).$$

Various applications of these identities are given. *L. Carlitz* (Durham, N. C.).

Kössler, M. Über ein Teilerproblem. Věstník Královské České Společnosti Nauk. Třída Matemat.-Přírodověd. 1943, 18 pp. (1943). [MF 16113]

Let $S_t(N, s) = \sum_1^N \sigma_t(k) k^{-s}$, where $\sigma_t(k)$ is the sum of the tth powers of the divisors of k. Making use of the second identity of another paper [see the preceding review], a rather complicated asymptotic formula for $S_t(N, s)$ is derived. *L. Carlitz* (Durham, N. C.).

Kesava Menon, P. Some generalizations of the divisor function. J. Indian Math. Soc. (N.S.) **9**, 32–36 (1945). [MF 15924]

Typical of the functions considered is the following. Let $\tau(M_1, \cdots, M_r)$ denote the number of sets of divisors $d_1, \cdots, d_r$ of $M_1, \cdots, M_r$, respectively, such that the greatest common divisor of $d_1, \cdots, d_r$ is 1. Then $\tau(M_1, \cdots, M_r)$ is multiplicative, that is,

$$\tau(M_1N_1, \cdots, M_rN_r) = \tau(M_1, \cdots, M_r)\tau(N_1, \cdots, N_r),$$

where $(\prod M_i, \prod N_i) = 1$; and similarly for $\tau'(M_1, \cdots, M_r)$, which is defined as the number of sets of divisors $d_1, \cdots, d_r$, every two of the d's being relatively prime. *L. Carlitz.*

Kesava Menon, P. On the equation $x_1^3 + x_2^3 = y_1^3 + y_2^3$. Math. Student **13**, 52–54 (1945). [MF 15643]

The general solution of the Diophantine equation

$$(1) \qquad x_1^3 + x_2^3 = y_1^3 + y_2^3$$

in rational numbers x_1, x_2, y_1, y_2 was found by Euler and simplified by Binet. This solution can be written in the form

$$x_1 = c\{(a+3b)(a^2+3b^2)-1\}, \quad x_2 = c\{-(a-3b)(a^2+3b^2)+1\},$$
$$y_1 = c\{(a^2+3b^2)^2-(a-3b)\}, \quad y_2 = c\{-(a^2+3b^2)^2+(a+3b)\},$$

where a, b, c are arbitrary rational numbers. Since the right-hand sides are not of the same degree, integral solutions exist for which the corresponding values of a and b are not integers. Therefore it is of interest to find a solution where the right-hand sides are of the same degree. Ramanujan [Hardy and Wright, An Introduction to the Theory of Numbers, Oxford University Press, 1938, p. 201] gave the particular solution

$$x_1 = 3a^2+5ab-5b^2, \quad x_2 = 4a^2-4ab+6b^2,$$
$$y_1 = 3b^2+5ab-5a^2, \quad y_2 = 4b^2-4ab+6a^2.$$

The author generalizes this result by proving the following theorem. Let a_1, a_2, b_1, b_2 be rational numbers satisfying the conditions that $a_1^3+a_2^3+b_1^3+b_2^3=0$ and $-(a_1+b_1)/(a_2+b_2)$ is the square of a rational number d. Then

$$x_1 = a_1a^2 \pm (a_2-b_2)ab/d - b_1b^2, \quad x_2 = a_2a^2 \pm (a_1-b_1)abd - b_2b^2,$$
$$y_1 = a_1b^2 \pm (a_2-b_2)ab/d - b_1a^2, \quad y_2 = a_2b^2 \pm (a_1-b_1)abd - b_2a^2$$

is a solution of (1), but not the general solution. For instance, we may choose $a_1=1$, $a_2=12$, $b_1=-9$, $b_2=-10$. From each such set a_1, a_2, b_1, b_2, infinitely many other such sets may be obtained. *A. Brauer* (Chapel Hill, N. C.).

✱Streefkerk, Hendrik. Over het Aantal Oplossingen der Diophantische Vergelijking $U = \sum_{i=1}^s (Ax_i^2+Bx_i+C)$. [On the Number of Solutions of the Diophantine Equation $U = \sum_{i=1}^s (Ax_i^2+Bx_i+C)$]. Thesis, Free University of Amsterdam, 1943. iv+102 pp. (Dutch)

The problem referred to in the title ($A>0$, B and C given integers) is equivalent in an elementary way to the determination of the number $r_s(M)$ of representations of a given number M ($M \equiv b^2s \pmod{4a}$; $(a, b)=1$) as a sum of s squares of integers $y \equiv b \pmod{2a}$. For $s \geqq 3$, the singular series $\rho_s(M)$ belonging to this problem is constructed (for $s=3$ and 4 it is obtained formally by analytic continuation) and summed in finite form. For $s \geqq 3$, the author systematically investigates when $r_s(M) = \rho_s(M)$ holds identically in M [cf. Hardy, Trans. Amer. Math. Soc. **21**, 255–284 (1920)]. The only cases appear to be $a=1$, $3 \leqq s \leqq 8$; $a=2$, $3 \leqq s \leqq 8$; $a=3$, $s=3$. The last case is new. It furnishes formulas for the number of representations of a given integer as a sum of 3 generalized pentagonal numbers ($\frac{3}{2}x^2 - \frac{1}{2}x$ with integral x), or of 3 generalized octagonal numbers ($3x^2-2x$ with integral x). The other cases lead to known results concerning sums of 3 to 8 squares or 3 to 8 triangular numbers. *N. G. de Bruijn* (Eindhoven).

de Bruijn, N. G. On the number of solutions of the system $x_1^2+x_2^2+x_3^2=n$, $x_1+x_2+x_3=m$. Nieuw Arch. Wiskunde (2) **22**, 53–56 (1943). (Dutch) [MF 15699]

The author gives an elegant proof of the (known) theorem that this number of solutions is $\alpha_N F(N)$, where $N = \frac{1}{2}(3n-m^2)$, $\alpha_N = 6$ or 3 according as $N \equiv 0$ or $\not\equiv 0 \pmod 3$, and where $F(N)$ is the sum $\sum(-3|d)$, summed over all divisors d of N. *H. D. Kloosterman* (Leiden).

Vandiver, H. S. On classes of Diophantine equations of higher degrees which have no solutions. Proc. Nat. Acad. Sci. U. S. A. **32**, 101–106 (1946). [MF 16362]

In many cases the impossibility of a Diophantine equation may be shown by the use of congruences with respect to a convenient modulus. The author uses this principle for constructing general classes of Diophantine equations which have no solutions or no nonzero solutions. His main result is the following. If c is a prime and m is an integer such that $p=1+mc$ with p prime, then

$$a_1x_1^m + a_2x_2^m + \cdots + a_sx_s^m = 0$$

has only the solution $x_1=x_2=\cdots=x_s=0$ provided that: $s \leqq c-2$; the sum of no n of the integers a is zero, $0<n\leqq s$; and

$$\left(\sum_{i=1}^s |a_i|\right)^{\varphi(c)} < p.$$

The condition that the sum of no n of the a's is zero is necessary since, if $a_1+a_2+\cdots+a_j=0$, the Diophantine equation is satisfied with $x_1=x_2=\cdots=x_j=1$ and $x_{j+1}=x_{j+2}=\cdots=x_s=0$. *T. Nagell* (Uppsala).

Beeger, N. G. W. H. A list of errors in a table of numbers D for which $x^2-Dy^2=-1$ has solutions in integers. Nieuw Arch. Wiskunde (2) **21**, 194–196 (1943). [MF 15703]

The tables referred to in the title are tables I and II of N. Nielsen [Recherches Numériques sur Certaines Formes Quadratiques, Copenhagen, 1931], which extend to $D=1000$. A complete list of twenty errata is given. The results of comparing these tables with those of P. Seeling [Arch. Math. Phys. **52**, 40–49 (1871)] and W. Patz [Tafel der regelmässigen Kettenbrüchen für Quadratwurzeln aus den natürlichen Zahlen 1–10000, Leipzig, 1941] are also given. *D. H. Lehmer* (Berkeley, Calif.).

Estermann, T. On the sign of the Gaussian sum. J. London Math. Soc. **20**, 66–67 (1945). [MF 16696]

This paper gives a simple determination, by means of an elementary estimate, of the sign in the formula for the Gaussian sum $S = \sum_{t=0}^{k-1} e^{2\pi i n^2/k}$ (k an odd integer greater than 1). *H. W. Brinkmann* (Swarthmore, Pa.).

Yamada, Kaneo. Berichtigung zu der Note: Eine Bemerkung zum Fermatschen Problem. Tôhoku Math. J. 48, 193–198 (1941). [MF 16359]

Corrections of erroneous [Zentralblatt für Math. 23, 8 (1941)] congruences of a former paper [same J. 45, 249–251 (1939)], relating to Fermat's quotient. The reviewer has found the new congruences to be correct [the factor p in (9) must be cancelled]. The author also gives a proof of the theorem of Gottschalk [Math. Ann. 115, 157–158 (1938)] that if, for the odd prime p, $mp=a\pm b$, $(p, m)=1$, and a and b are not divisible by primes other than 2, 3, 5, 11, 17 (if $p=6n-1$ by no prime greater than 19), then $x^p+y^p+z^p=0$, $(xyz, p)=1$ has no solution. Let $Q(a)\equiv(a^{p-1}-1)/p \pmod{p}$; from the congruences of Eisenstein,

$$Q(g)+Q(h)\equiv Q(gh) \pmod{p},$$
$$Q(mp\pm b)\equiv Q(b)\mp m/b \pmod{p^2},$$

it follows that $Q(2)\equiv Q(3)\equiv Q(5)\equiv\cdots\equiv 0 \pmod{p}$ is impossible; hence, from theorems of Wieferich, Mirimanoff and others, that $x^p+y^p+z^p=0$ is impossible with $(xyz, p)=1$.

N. G. W. H. Beeger (Amsterdam).

Banerjee, D. P. On a theorem in the theory of partition. Bull. Calcutta Math. Soc. 37, 113–114 (1945). [MF 15686]

The author gives a simple proof of the following theorem due essentially to Sylvester [Collected Mathematical Papers, v. 3, Cambridge University Press, 1909, pp. 680–686]. Let $p(n)$ denote the number of unrestricted partitions of n and $Q(n)$ the number of partitions of n into parts which are both odd and distinct. Then $p(n)-Q(n)$ is even. A small table of $Q(n)$ for $n\leqq 70$ is given. [For a larger table of $Q(n)$ ($n\leqq 400$) see G. N. Watson, Proc. London Math. Soc. (2) 42, 550–552 (1937).] *D. H. Lehmer.*

Iseki, Kaneshirô. Ein Theorem der Zahlentheorie. Tôhoku Math. J. 48, 60–63 (1941). [MF 16347]

The author denotes by $\rho_k(n)$ the number of partitions of n into k positive summands, two summands being counted the same if they differ only in order. He proves that, for each k,

$$\rho_k(n)\sim n^{k-1}/k!(k-1)!, \qquad n\to\infty.$$

The proof is elementary and the results closely related to those of Erdös and Lehner [Duke Math. J. 8, 335–345 (1941); these Rev. 3, 69]. *B. W. Jones* (Ithaca, N. Y.).

Reitan, L. On solutions of the number theoretical equation $x^2+y^2+z^2+v^2=t$ for a given t. Norsk Mat. Tidsskr. 28, 21–23 (1946). (Norwegian) [MF 16694]

Salzer, Herbert E. On numbers expressible as the sum of four tetrahedral numbers. J. London Math. Soc. 20, 3–4 (1945). [MF 15926]

The author conjectures two theorems. (a) Every square is the sum of less than 5 tetrahedral numbers $\frac{1}{6}n(n+1)(n+2)$ ($n>0$). (b) Every tetrahedral is the sum of d tetrahedrals with $2\leqq d\leqq 4$. They have been verified by the author in the first 200 cases. If in (a) "square" is replaced by "triangular" the first 200 cases hold except for the triangular number 153, which requires 5 tetrahedrals.

Pollock's conjecture that every integer is the sum of less than 6 tetrahedrals is verified up to 1000. In fact, 4 tetrahedrals suffice with 45 exceptions: 17, 27, 33, $\cdots$, 953.

D. H. Lehmer (Berkeley, Calif.).

Suryanarayana Rao, B. On numbers which are the sum or difference of two cubes. Math. Student 13, 57–58 (1945). [MF 15645]

The author gives a parametric representation of the integers which are the sum of two cubes. *A. Brauer.*

Srinivasan, A. K. Residual types of partitions of "0" into four cubes. Math. Student 13, 47–48 (1945). [MF 15641]

Elementary remarks. *A. Brauer* (Chapel Hill, N. C.).

Souriau, Jean-Marie. Généralisation de certaines formules arithmétiques d'inversion. Applications. Revue Sci. (Rev. Rose Illus.) 82, 204–211 (1944). [MF 16272]

The author generalizes the functions of Möbius and Euler. Let α be an arbitrary complex number and set

$$\eta_\alpha(n)=(\alpha+1)\cdots(\alpha+n-1)/n!, \quad n\geqq 1;\ \eta_\alpha(0)=1.$$

If $n=p_1^{\lambda_1}p_2^{\lambda_2}\cdots p_k^{\lambda_k}$, set

$$\mu_\alpha(n)=\eta_\alpha(\lambda_1)\eta_\alpha(\lambda_2)\cdots\eta_\alpha(\lambda_k), \qquad \mu_\alpha(1)=1.$$

For $\alpha=-1$ we have Möbius's function $\mu(n)$. If s is arbitrary complex, set

$$\varphi_\alpha^s(n)=\sum_{k|n}\mu_\alpha(k)(n/k)^s.$$

For $s=1$, $\alpha=-1$ we get Euler's function $\varphi(n)$. If p and q are relatively prime, $\mu_\alpha(pq)=\mu_\alpha(p)\mu_\alpha(q)$. Furthermore,

$$\sum_{k|n}\mu_\alpha(k)\mu_\beta(n/k)=\mu_{\alpha+\beta}(n),$$

which contains a number of special interesting cases.

The author's discussion of arithmetical inversion formulas is based on the introduction of operator families $\{T_\alpha\}$ which form a group of transformations isomorphic to the additive group on the subscript α. A typical case is given by the operator

$$S_\alpha[f(n)]=\sum_{k|n}\mu_\alpha(k)f(n/k), \quad S_0[f(n)]=f(n),$$

with $S_\alpha\{S_\beta[f(n)]\}=S_{\alpha+\beta}[f(n)]$. The author proves that for each α we have $\mu_\alpha(n)=o(n^\epsilon)$ for every $\epsilon>0$. He also discusses the power series $f_\alpha(x)=\sum_1^\infty\mu_\alpha(n)x^n$, $|x|<1$, which satisfies

$$\sum_1^\infty\mu_\beta(k)f_\alpha(x^k)=f_{\alpha+\beta}(x),$$

and the Dirichlet series $\sum\mu_\alpha(n)n^{-s}=[\zeta(s)]^\alpha$, together with related mean values. *E. Hille* (New Haven, Conn.).

Turán, P. Über die Verteilung der Primzahlen. I. Acta Univ. Szeged. Sect. Sci. Math. 10, 81–104 (1941). [MF 15836]

It was proved by Hoheisel [S.-B. Preuss. Akad. Wiss. Phys.-Math. kl. 1930, 580–588] that $\pi(x+x^\theta)-\pi(x)\sim x^\theta/\log x$ as $x\to\infty$, where $\pi(x)$ is the number of primes not exceeding x and $\theta=32999/33000$. Heilbronn, Tchudakoff, and Ingham in turn reduced the value of θ. Ingham's main result [Quart. J. Math., Oxford Ser. 8, 255–266 (1937)] is that if $\zeta(\frac{1}{2}+it)=O(t^c)$ as $t\to\infty$ then $N(\sigma, T)=O(T^{2(1+2c)(1-\sigma)}\log^5 T)$ for $\frac{1}{2}<\sigma\leqq 1$ as $T\to\infty$. Here $N(\sigma, T)$ is the number of zeros $\beta+i\gamma$ of $\zeta(s)$ with $\beta\geqq\sigma$, $0<\gamma\leqq T$. His value of θ is then $(1+4c)/(2+4c)+\epsilon$ for every $\epsilon>0$. In the present paper the author proves, subject to a certain hypothesis, that

$$N(\sigma, T)\leqq T^{2(1-\sigma)}\exp(13\log^{0.18}T),$$

uniformly for $\frac{1}{2}\leqq\sigma\leqq 1$, $T\geqq C$ and

$$\pi(x+\sqrt{x}\exp(\log^{0.996}x))-\pi(x)\sim\sqrt{x}\exp(\log^{0.996}x)/\log x,$$

which are improvements on Ingham's results. The point of the paper is that the new conjecture, unlike the Riemann or Lindelöf hypotheses, has apparently no direct connection with the prime numbers. The author's hypothesis reads as follows: if $n\geqq C$ and $z_1=1$, $|z_2|\leqq 1, \cdots, |z_n|\leqq 1$, then the maximum value of $|z_1^\nu+z_2^\nu+\cdots+z_n^\nu|$ for $n^{1.5}-n^{1.09}\leqq\nu\leqq n^{1.5}$ is greater than $\exp(-n^{0.09})$. It is used to approximate a sum which arises in connection with the integral

$$J=\int_{T/2}^{T}|g_\nu(s)|^2dt,$$

where $s=\sigma_0+it$ and $g_\nu(s)$ is the $(\nu+1)$th derivative of $\log\zeta(s)$. The study of the integral is suggested by the following considerations: (a) with suitable restrictions on $s_0=\sigma_0+it_0$ the function $\zeta(s)$ has no zeros in the circle

$$|s-s_0|=\liminf\left[|g_\nu(s_0)|/\nu!\right]^{-1/\nu}=R;$$

(b) an inequality $|g_\nu(s)|\leqq T^{\epsilon-1}\sqrt{J}$ holds for $T/2\leqq t<T$ with the exception of a set of intervals of total length less than $c(\epsilon)T^{2(1-\sigma)+\epsilon}$. In contrast to classical methods the author works with functional values away from the critical strip. *R. D. James* (Vancouver, B. C.).

Wintner, Aurel. The fundamental lemma in Dirichlet's theory of the arithmetical progressions. Amer. J. Math. 68, 285–292 (1946). [MF 16425]

Let $\chi(n)$ be any multiplicative function (that is, $\chi(mn)=\chi(m)\chi(n)$ for all positive integers m and n) which satisfies the restriction (*) $\chi(p)\geqq -1$ for every prime p. Suppose also that the Dirichlet series $J(s)=\sum_{n=1}^{n=\infty}\chi^2(n)n^{-s}$ is convergent for $s>1$. Then so, clearly, is $L(s)=\sum_{n=1}^{n=\infty}\chi(n)n^{-s}$. The author proves that, if $L(s)$ admits across the point $s=1$ an analytic continuation which is regular for $\frac{1}{2}\leqq s\leqq 1$, then $L(1)\neq 0$. This result is a generalisation of Dirichlet's well-known result for a real nonprincipal character $\chi(n)$, and also of a more general result due to Ingham in which (in the case when χ is real) $\chi(p)$ is restricted to the values 0, ± 1, this restriction being essential to the method of proof [A. E. Ingham, J. London Math. Soc. 5, 107–112 (1930)]. Landau's theorem on the singularities of a Dirichlet series with positive coefficients occupies a central position in the author's proof, as also in the work of Ingham and earlier writers, and is applied to the function

$$\sum_{n=1}^{\infty}\beta(n)n^{-s}=\zeta(s)L(s)/J(2s),$$

whose coefficients $\beta(n)$ are nonnegative because of (*). If $L(1)=0$, it can be concluded that this series is convergent for $s>\frac{1}{2}$, and this, combined with the assumption regarding the behaviour of $L(s)$ in the neighbourhood of $s=\frac{1}{2}$, eventually produces a contradiction. The author's result raises the interesting question whether the inequality (*) is the best possible. Whether this is so is not known, although the manner in which (*) arises in the proof leaves little doubt that it is. *R. A. Rankin* (Cambridge, England).

Wintner, Aurel. A factorization of the densities of the ideals in algebraic number fields. Amer. J. Math. 68, 273–284 (1946). [MF 16424]

Let $F(n)$ be a multiplicative function of the positive integer n (so that $F(mn)=F(m)F(n)$ whenever m and n are relatively prime) for which $f(s)=\sum_{n=1}^{\infty}F(n)/n^s$, where $s=\sigma+it$, is absolutely convergent for $\sigma>1$ and represents there a function with a simple pole at $s=1$. The author proves that, if $|F(p^k)|<Kp^{\theta(k-1)}$ for some fixed K and some fixed $\theta<1$ and for every prime power p^k, the infinite series $\sum(F(p)-1)/p$ and the infinite product

$$\prod(1-p^{-1})\left(1+\sum_{k=1}^{\infty}F(p^k)p^{-k}\right)$$

are convergent. Here p runs through all primes in increasing order. The value of the infinite product is the residue of $f(s)$ at $s=1$. If the assumption about F is satisfied for every $k\neq 2$, but is relaxed from $F(p^2)=O(p^\theta)$ to $F(p^2)=O(p)$ for $k=2$, then the infinite product may become divergent or it may converge to a value distinct from the residue. The author deduces from the theorem that the residue at $s=1$ of the zeta-function $\zeta(s;\mathfrak{K})$ of an algebraic number field (which is identical with the asymptotic density of the integral ideals of $\mathfrak{K}$) can be evaluated in the form

$$\prod(1-p^{-1})(1-p^{-g_1})^{-1}(1-p^{-g_2})^{-1}\cdots(1-p^{-g_j})^{-1},$$

where p runs through all rational primes in increasing order; $j=j(p)$ is the number of distinct prime ideals which divide p and the g_i are the respective degrees of these j prime ideals. In an appendix the author considers the nature of analytical limitations imposed on the zeta-function by the laws of factorization in the field $\mathfrak{K}$. *H. D. Kloosterman.*

Erdös, P. On the distribution function of additive functions. Ann. of Math. (2) 47, 1–20 (1946). [MF 15655]

Let $f(n)$ be an additive function of the positive integer n, that is, $f(m_1m_2)=f(m_1)+f(m_2)$ if $(m_1, m_2)=1$. Let p be an arbitrary prime and let $f'(p)=f(p)$ if $|f(p)|\leqq 1$ and $f'(p)=1$ otherwise. The author shows that, if $\sum(f'(p))^2/p=\infty$ and if $f(p)\rightarrow 0$ as $p\rightarrow\infty$, then the sequence $f(n)-[f(n)]$, $n=1, 2, \cdots$, is equidistributed on the interval $[0, 1]$. The methods are similar to those used by Erdös and Kac [Amer. J. Math. 62, 738–742 (1940); these Rev. 2, 42] and depend on a strengthening of the central limit theorem due to Berry [Trans. Amer. Math. Soc. 49, 122–136 (1941); these Rev. 2, 228].

Most of the other theorems in the paper depend on the following result. There exists a constant c such that, if $f^+(n)$ is defined by $f^+(n)=f(n)-c\log n$, then $\sum(f^{+\prime}(p))^2/p<\infty$ if and only if there exist two positive constants c_1, c_2 such that, for infinitely many n, there are integers $a_1<a_2<\cdots<a_x(<n)$ for which $|f(a_i)-f(a_j)|<c_1$ and $x>c_2n$. The proof of "only if" depends on the device of estimating $\sum_{m=1}^{n}(f(m))^2$ by splitting the corresponding double sum at $\sqrt{n}$. The proof of "if" is more difficult and depends on methods used by Turán [J. London Math. Soc. 11, 125–133 (1936)] and by Erdös [J. London Math. Soc. 12, 9–10 (1937)]. This theorem leads to the following characterizations of the additive function $f(n)=c\log n$: $f(n)=c\log n$ if and only if $f(n+1)\geqq f(n)$ holds for all n; also, if and only if $f(n+1)-f(n)\rightarrow 0$ as $n\rightarrow\infty$. The author makes several conjectures along these lines. He also deduces, from the main theorem, a generalization of his theorem concerning the continuity of the asymptotic distribution of additive functions [J. London Math. Soc. 13, 119–127 (1938), in particular, p. 121]. If $\sum_{f(p)\neq 0}1/p=\infty$ and if $a_1<a_2<\cdots<a_x\leqq n$ are integers such that $|f(a_i)-f(a_j)|<\epsilon$, then $x<\delta n$, where $\delta=\delta(\epsilon)\rightarrow 0$ as $\epsilon\rightarrow 0$.

The paper contains other related theorems. A number of annoying misprints occur: the statement of theorem II should involve $\phi(m)$, not $f(m)$; the statement "$\delta\rightarrow 0$ as $\epsilon\rightarrow 0$" should be included in theorem IV; $\sum_p(f^{+\prime}(p))^2/p$ should replace $\sum_p(f^{+\prime}(p)/p)^2$ in theorem V; lemma 7 is not accurately stated, etc. *P. Hartman* (Flushing, N. Y.).

Wang, Fu Traing. A note on the Riemann zeta-function. Bull. Amer. Math. Soc. **52**, 319–321 (1946). [MF 16197]

Let $\rho_\nu = \beta_\nu + i\gamma_\nu$ be the zeros of $\zeta(\frac{1}{2}+z)$, with $\beta_\nu \geqq 0$, $\gamma_\nu > 0$. It is proved by means of contour integration of $z^{-2}\log\zeta(\frac{1}{2}+z)$ [cf. Titchmarsh, The Zeta-Function of Riemann, Cambridge University Press, 1930, p. 53] that

$$(1)\quad \int_1^T t^{-2}\log|\zeta(\tfrac{1}{2}+it)|\,dt = 2\pi\sum_1^\infty \beta_\nu|\rho_\nu|^{-2} + A + O(T^{-1}\log T),$$

where

$$A = \int_0^{\pi/2} \Re\{e^{-i\theta}\log\zeta(\tfrac{1}{2}+e^{i\theta})\}\,d\theta.$$

Consequently, a necessary and sufficient condition for the truth of the Riemann hypothesis is that

$$\int_1^\infty t^{-2}\log|\zeta(\tfrac{1}{2}+it)|\,dt = A.$$

[The O-term in (1) can be replaced by $O(T^{-1}\log\log T)$.]
N. G. de Bruijn (Eindhoven).

Selberg, Atle. On the remainder in the formula for $N(T)$, the number of zeros of $\zeta(s)$ in the strip $0<t<T$. Avh. Norske Vid. Akad. Oslo. I. **1944**, no. 1, 27 pp. (1944). [MF 16438]

The formula considered is

$$N(T) = (1/2\pi)T\log(T/2\pi e) + \tfrac{7}{8} + S(T) + O(1/T)$$

and it is a question of proving theorems about the term $S(T)$. The main result is that, on the Riemann hypothesis, for every fixed positive integer k

$$\int_0^T |S(t)|^{2k}dt = \frac{(2k)!}{k!(2\pi)^{2k}}\,T(\log\log T)^k + O\{T(\log\log T)^{k-1}\}.$$

There are similar results with integrals over $(T, T+H)$, and also when $S(t)$ is replaced by its integral of any order. Applications are made to questions of changes of sign of $S(t)$.
E. C. Titchmarsh (Oxford).

Popken, J. An arithmetical property of a class of Dirichlet's series. Nederl. Akad. Wetensch., Proc. **48**, 517–534 = Indagationes Math. **7**, 105–122 (1945). [MF 15804]

The purpose of the paper is to prove the following theorem. Let the convergent Dirichlet series $\sum_{k=1}^\infty a_k e^{-\lambda_k s}$ $(\lambda_1<\lambda_2<\cdots;\ \lim\lambda_k=\infty)$ represent an analytic function $f(s)$ satisfying an algebraic differential equation

$$P(s, f(s), f'(s), \cdots, f^{(n)}(s)) = 0,$$

P being a polynomial in $s, f', \cdots, f^{(n)}$. Let the coefficients a_k be rational and let p_k denote the largest prime divisor of the denominator of a_k written in its irreducible form. Then there exists a positive number c such that $p_k < \lambda_k^c$ except for a finite number of values of k. Transforming the power series with rational coefficients $\sum a_k x^k$ into a Dirichlet series by the substitution $x = e^{-s}$, the author obtains as a special case of his theorem the well-known result of Hurwitz that $p_k < h^{c_1}$ $(h>1)$, where c_1 is a constant. *R. Salem.*

Ollerenshaw, Kathleen. Lattice points in a hollow n-dimensional hypercube. J. London Math. Soc. **20**, 22–26 (1945). [MF 15930]

For positive μ let H_μ be the hypercube $|x_r|\leqq\mu$, $r=1,\cdots,n$, in Euclidean n-space. For $\mu<1$ let F_μ be the set of points in H_1 which are not interior points of H_μ. Let $m=[1/(1-\mu)]$. The author shows that a point of every lattice with determinant not greater than $(1/m)^n$, other than the origin, exists in F_μ. If $\mu<\frac{1}{2}$ and H_μ contains a lattice point other than the origin then F_μ is shown to contain an interior lattice point and so the result is equivalent to Minkowski's linear form theorem for H_1. For $\mu\geqq\frac{1}{2}$ the lattice point in F_μ is interior to it except when the lattice assumes the special form $x_r=\xi_r/m$, $r=1,\cdots,n$, while $\xi_1,\cdots,\xi_n$ run through all integers. The result is a consequence of the recently proved Minkowski conjecture on the boundary case of the linear form theorem [G. Hajós, Math. Z. **47**, 427–467 (1941); these Rev. **3**, 302]. A lattice point theorem for an arbitrary lattice is obtained by a magnification of the n-space. *D. Derry* (Vancouver, B. C.).

Jarník, Vojtěch. Zwei Bemerkungen zur Geometrie der Zahlen. Věstník Královské České Společnosti Nauk. Třida Matemat.-Přírodověd. **1941**, 12 pp. (1941). (Czech. German summary) [MF 16118] no. 2(?)

Proofs of the following theorems. (1) Let N be a bounded closed set in n-dimensional Euclidean space with at least one inner point, $V(N)$ the set of all points $X-Y$, where X, Y lie in N, and $J(N)$ the inner Jordan measure of N. Denote by τ_1 the smallest positive number such that $\tau_1V(N)$ contains a lattice point $X_1\neq0$ and, for $k=2,3,\cdots,n$, by τ_k the smallest positive number such that $\tau_kV(N)$ contains a lattice point X_k independent of $X_1,\cdots,X_{k-1}$. Then $\tau_1\tau_2\cdots\tau_nJ(N)\leqq2^{n-1}$. The author states that V. Knichal has proved that the constant 2^{n-1} cannot, in general, be replaced by 1; for convex bodies N, this is, however, allowed by Minkowski's classical theorem [Geometrie der Zahlen, Leipzig, 1896–1910, § 53; see also H. Davenport, Quart. J. Math., Oxford Ser. **10**, 119–121 (1939)]. (2) Let now M be a convex body and $N=\frac{1}{2}M$, so that $V(N)=M$. Denote by σ the smallest positive number such that the point set $X+\sigma M$ contains for every point X at least one lattice point. Then $\frac{1}{2}\tau_n\leqq\sigma\leqq(n/2)\tau_n$, and this is the best possible result. (3) Let $L_1(X),\cdots,L_n(X)$ be n real linear forms in $x_1,\cdots,x_n$ of determinant 1, and let $\epsilon>0$. Then there exists a lattice point $X\neq0$ such that $|L_1(X)\cdots L_n(X)|\leqq2^{-(n-1)/2}+\epsilon$. [This is not the best possible result; in an as yet unpublished paper, H. Davenport has shown that approximately 12^{-n} may be taken as the right-hand side.] *K. Mahler.*

Jarník, Vojtěch. Zur Gitterpunktlehre der Ellipsoide $\alpha_1(u_1^2+\cdots+u_{r_1}^2)+\alpha_2(u_{r_1+1}^2+\cdots+u_r^2)\leqq x$. 1940, no. 3, 63 pp. ~~1940, 63 pp.~~ Věstník Královské České Společnosti Nauk. Třida Matemat.-Přírodověd. **1940**, 63 pp. (1940). (German. Czech and French summaries) [MF 16126]

Let r_1, r_2 be integers, $\alpha_1>0$, $\alpha_2>0$, α_1/α_2 irrational, $r=r_1+r_2$, $z=\min(r_1,r_2)\geqq6$. Put

$$Q(u)=\alpha_1(u_1^2+\cdots+u_{r_1}^2)+\alpha_2(u_{r_1+1}^2+\cdots+u_r^2),$$

and let $A_Q(x)$ be the number of lattice points in the ellipsoid $Q(u)\leqq x$, and V_Q the volume of $Q(u)\leqq1$. Write

$$\mathrm{P}_Q(x)=A_Q(x)-V_Qx^{\frac{1}{2}r},\quad M_Q(x)=\int_0^x \mathrm{P}_Q^2(y)dy,\quad x\geqq0.$$

Denote by p_v/q_v $(v=0,1,\cdots)$ the continued fraction convergents to α_1/α_2, and put

$$F_Q(x)=x^{r-1}\sum_{v,m,n} q_v^{-2}n^{2-r_1}m^{2-r_2}\min\{1,(q_{v+1}mnx^{-1})^2\},$$

the summation being carried out over all nonnegative integers v, m, n satisfying $p_v>0$, $m|p_v$ and $n|q_v$. The greater part of the paper is devoted to the proof of the principal theorem which states that, if $0\leqq\mu<1$, there exist positive

numbers c_1, c_2, c_3 depending only on μ, r_1, r_2, α_1, α_2 such that $c_1F_Q(x)<M_Q(x)-M_Q(\mu x)<c_2F_Q(x)$ for all $x>c_3$. A considerable amount of information on the range of variation of $M_Q(x)$ for different α_1, α_2 can be deduced from this result (with $\mu=0$). Let (for $x\rightarrow\infty$)

$$f_1(Q)=\overline{\lim}\ \{\log M_Q(x)/\log x\},$$
$$f_2(Q)=\underline{\lim}\ \{\log M_Q(x)/\log x\},$$
$$\gamma=\gamma(\alpha_1, \alpha_2)=\overline{\lim}\ \{\log q_{\nu+1}/\log q_\nu\},$$
$$\delta=\delta(\alpha_1, \alpha_2)=\underline{\lim}\ \{\log q_{\nu+1}/\log q_\nu\},$$

and let $f_3(r_1, r_2, \gamma)$ be the upper bound of $f_2(Q)$ for given r_1, r_2, γ. Clearly $1\leqq\delta\leqq\gamma$ and $f_2(Q)\leqq f_1(Q)$. Write $d=\max\{0, 2-|r_1-r_2|\}$, $\beta=r-1-2s/(s+2)$ and, finally, $\beta'=r-1-2s/(s+2+d/\gamma)$.

Of the various results proved the following are the most important. (i) $f_1(Q)=r-1-2/\gamma\geqq r-3$. (This can also be deduced from a previous paper of the author [Math. Z. 36, 581–617 (1933)].) (ii) There exist functions W_1 and W_2 (of forms too complicated to quote here) depending only on s, γ, d such that $W_1<2$ $(\gamma>1)$, $W_1=2$ $(\gamma=1)$, and $r-1-W_1\leqq f_3(r_1, r_2, \gamma)\leqq r-1-W_2$. Furthermore, corresponding to every set r_1, r_2, γ there exists a form Q with $f_2(Q)\geqq r-1-W_1$. From (ii) it follows easily that (a) $f_2(Q)<f_1(Q)$ for $\gamma>1$; (b) $f_3(r_1, r_2, \gamma)=\beta'$ for $\gamma>s+1$; (c) for given r_1, r_2 a form Q exists with $f_2(Q)\geqq\beta$; and (d) for given r_1, r_2 with $|r_1-r_2|>1$, $f_2(Q)\leqq\beta$ and there exists a form with $f_2(Q)=\beta$. (iii) If $\delta>s+2$, $f_2(Q)=\beta'$, and if, in addition, $|r_1-r_2|\geqq2$, then $\underline{\lim}\ x^{-\beta}M_Q(x)>0$. (iv) If $|r_1-r_2|\geqq4$, $\underline{\lim}\ x^{-\beta}M_Q(x)<\infty$.

R. A. Rankin (Cambridge, England).

Rédei, L. Zur Gaussischen Theorie der Reduktion binärer quadratischer Formen. Acta Univ. Szeged. Sect. Sci. Math. 10, 134–140 (1941). [MF 15840]

The author gives a simple proof of Gauss's theorem that equivalent reduced binary quadratic forms with positive determinant belong to the same period. The proof is by a method, not using continued fractions, somewhat similar to that of F. Mertens [J. Reine Angew. Math. 89, 332–339 (1880)]. A method is deduced for finding the fundamental solution of Pell's equation $t^2-Du^2=1$ without using continued fractions. *H. S. A. Potter* (Aberdeen).

Davenport, H. The reduction of a binary cubic form. I. J. London Math. Soc. 20, 14–22 (1945). [MF 15929]

Let $f(x, y)=ax^3+bx^2y+cxy^2+dy^3$, where the coefficients are real and the discriminant D is positive. A simple arithmetical proof is given of Mordell's theorem [same J. 18, 201–217 (1943); these Rev. 6, 37] that there exist integers x, y not both zero for which $|f(x, y)|\leqq(D/49)^{1/4}$, and this with strict inequality unless $f(x, y)$ is equivalent to

$$(D/49)^{1/4}(x^3+x^2y-2xy^2-y^3).$$

The author also proves that, if $f(x, y)$ is reduced and $D=49$, then at least one of the products $f(1, 0)f(0, 1)$, $f(1, 0)f(1, 1)$, $f(1, 0)f(1, -1)$, $f(0, 1)f(1, 1)$, $f(0, 1)f(1, -1)$ does not exceed 1 numerically. One of them is numerically less than 1 except when $\pm f(x, y)=x^3+x^2y-2xy^2-y^3$ or

$$\pm f(x, y)=x^3+2x^2y-xy^2-y^3.$$

If $f(x, y)=0$ for integers x, y not both zero the above results are truisms. In this case the author proves that there is no upper bound in terms of D for a nonzero value of $f(x, y)$.

H. S. A. Potter (Aberdeen).

Davenport, H. The reduction of a binary cubic form. II. J. London Math. Soc. 20, 139–147 (1945).

[Cf. the preceding review.] The author defines a method of reduction for the binary cubic form with real coefficients $f(x, y)=ax^3+bx^2y+cxy^2+dy^3$ and negative discriminant $D=18abcd+b^2c^2-4ac^3-4db^3-27a^2d^2$. He proves that, if $f(x, y)$ is a reduced binary cubic form of discriminant -23, then one at least of $f(1, 0)$, $f(0, 1)$, $f(1, -1)$, $f(1, -2)$ does not exceed 1 numerically. Moreover, one of them is numerically less than 1 except when $f(x, y)=x^3+x^2y+2xy^2+y^3$ (in which case all four values are ±1). *H. S. A. Potter.*

Chabauty, Claude. Approximation des nombres algébriques et points pseudo-entiers des courbes algébriques. C. R. Acad. Sci. Paris 218, 899–901 (1944). [MF 15237]

Indications are given for a proof of the following theorem. Let $C:f(x, y)=0$ be a nonrational algebraic curve with at most two different points at infinity; let K be a finite algebraic field, α an integer in K, $\nu(\alpha)$ the number of different prime ideal factors of α. A constant $c>1$ independent of α exists such that there are at most $c^{\nu(\alpha)}$ points (x, y) on C, x and y being elements of K for which $\alpha^r x$, $\alpha^s y$ are integers in K if r, s are chosen suitably. *K. Mahler.*

ANALYSIS

***Pólya, G., und Szegö, G. Aufgaben und Lehrsätze aus der Analysis.** Dover Publications, New York, N. Y., 1945. Band I, xxiv+342 pp. \$3.50. Band II, xviii+412 pp. \$3.50.

Photographic reproduction of volumes 19 and 20 of the series Die Grundlehren der mathematischen Wissenschaften, published by J. Springer, Berlin, 1925. Some errors have been corrected and a more detailed index in German and English has been added.

Popoviciu, Tiberiu. On some inequalities. Gaz. Mat., București 51, 81–85 (1946). (Romanian) [MF 16555]

If $f(0)=0$, $f(x)$ is continuous for $x\geqq0$, and $(-1)^{n-1}f^{(n-1)}(x)$ is increasing for $x>0$, then for $x_k>0$ we have

$$\sum_{k=1}^{n}\{(-1)^{k-1}\sum{}^*f(x_{i_1}+\cdots+x_{i_k})\}>0,$$

where $\sum^*$ is summed over all combinations $i_1, \cdots, i_k$ of k of the integers $1, \cdots, n$. This inequality is illustrated by application to $f(x)=\log(x+1)$, $f(x)=\log\{(a^{x+1}-1)/(a-1)\}$, and hence to the number-theoretical functions $\sigma(n)$, $\sigma_m(n)$, $\varphi(n)$. The results are

$$\prod_{k=1}^{n}\{\prod{}^*\sigma_m(n_{i_1}n_{i_2}\cdots n_{i_k})^{(-1)^{k-1}}\}\geqq1,$$

where $\prod^*$ is defined analogously to $\sum^*$, and the reversed inequality for $\varphi(n)$. *R. P. Boas, Jr.* (Providence, R. I.).

Kreis, H. Arithmetisches und geometrisches Mittel. Elemente der Math. 1, 37–38 (1946). [MF 16377]

An inductive proof of the inequality between the means in question. *R. P. Boas, Jr.* (Providence, R. I.).

Cioranescu, Nicolas. Une nouvelle formule de moyenne integro-différentielle. Bull. Math. Soc. Roumaine Sci. 46, 107–112 (1944). [MF 16513]

Continuing his earlier work [Bull. École Polytech.

Bucarest [Bul. Politehn. Bucureşti] 12, 37–40 (1941); these Rev. 7, 115], the author considers functions f, g, φ, ψ of class C' in (a, b), φ and ψ being monotonic and similarly ordered; $p(x)$ is a nonnegative weight function and $M[f]=\int pfdx/\int pdx$. He obtains the formula

$$\frac{M[fg]-M[f]M[g]}{M[\varphi\psi]-M[\varphi]M[\psi]}=\frac{f'(\xi)g'(\xi_1)}{\varphi'(\xi)\psi'(\xi_1)},$$

where $a<\xi<b$, $a<\xi_1<b$. Specializations give inequalities between $M[f]$ and $M[1/f]$, a modified form of Taylor's formula with remainder, etc. *R. P. Boas, Jr.*

Mikusiński, Jean G. Sur l'inégalité différentielle $|f^{(n)}(x)| \geq m|f(x)|$. C. R. Acad. Sci. Paris 222, 359–361 (1946). [MF 16011]

If $|f^{(n)}(x)|\geqq m|f(x)|>0$ in $a<x<b$ and $f(a+)=f(b-)=0$, then $b-a<Nm^{1/n}$, where an admissible value for N is $\{1+\frac{1}{2}n(n+1)\}\{(n+1)!\}^{1/n}$. *R. P. Boas, Jr.*

Obrechkoff, Nikola. Quelques inégalités nouvelles sur les dérivées des fonctions. C. R. Acad. Sci. Paris 222, 531–533 (1946). [MF 16033]

Let $f(x)$ and $\varphi(x)$ have nth derivatives for $x>a$, $\varphi^{(n)}(x)\neq 0$, $|f^{(n)}(x)|\leqq|\varphi^{(n)}(x)|$. For a nonnegative integer m let $\lim x^{-m}f(x)=A$, $\lim x^{-m}\varphi(x)=B$, when $x\rightarrow\infty$ through some sequence $\{x_k\}$. Then $|f^{(m)}(x)-m!A|\leqq|\varphi^{(m)}(x)-m!B|$, $x>a$. This is obtained from a more general result in terms of divided differences of order m. *R. P. Boas, Jr.*

Mandelbrojt, Szolem. Sur les fonctions indéfiniment dérivables. C. R. Acad. Sci. Paris 222, 577–579 (1946). [MF 16034]

Let $f(x)$ be bounded and of class C^∞ in $0\leqq x<\infty$, satisfying $|f^{(\nu)}(x)|\leqq k^\nu M_\nu$, $f^{(p_n)}(0)=0$, where $p_n=0$, p_n increases and $\liminf n/p_n=D>\frac{1}{2}$. The author states conditions, too lengthy to quote here, under which necessarily $f(x)\equiv 0$. These generalize the Denjoy-Carleman theorem on quasi-analyticity and also the conditions given previously by the author [Trans. Amer. Math. Soc. 55, 96–131 (1944); these Rev. 5, 176]. *R. P. Boas, Jr.* (Providence, R. I.).

Reid, William T. Integral criteria for solutions of linear differential equations. Duke Math. J. 12, 685–694 (1945). [MF 15511]

The author gives a direct proof of the following generalisation of the fundamental lemma of the calculus of variations. Let $a_i(x)$, $i=0, 1, \cdots, n$, be of class $C^{(i)}$ for $a\leqq x\leqq b$, $a_n(x)\neq 0$. Let $M(y)=\sum_{i=0}^n(-1)^i\{a_i(x)y\}^{(i)}$ and let $f(x)$ and $g(x)$ be integrable. If $\int_a^b\{M(y)f(x)+y(x)g(x)\}dx=0$ for all $y(x)$ of class $C^{(n)}$ with $y^{(r)}(a)=y^{(r)}(b)=0$, $r\leqq n-1$, then $f(x)$ is equivalent to some function $u(x)$ satisfying $\sum_{i=0}^n a_i(x)u^{(i)}+g(x)=0$. By applying this theorem together with the properties of the pth integral mean

$$M_p[h;f(x)]=h^{-p}\int_x^{x+h}ds_p\int_{s_p}^{s_p+h}ds_{p-1}\cdots\int_{s_2}^{s_2+h}f(s_1)ds_1,$$

the following result is obtained. Let $a_i(x)$ be as before, $g(x)$ integrable on (a, b) and $f(x)$ integrable on (α, β) whenever $a<\alpha<\beta<b$. Then f is equivalent to some solution u of $\sum a_i(x)u^{(i)}+g(x)=0$ if and only if, for each such α, β,

$$\int_\alpha^\beta\left|\sum_{i=1}^n a_i(x)h^{n-i}\Delta_h^i f(x)+h^n\{a_0(x)f(x)+g(x)\}\right|dx=o(h^n).$$

Alternative proofs are given of the special case $a_n(x)\equiv 1$, $g(x)\equiv a_i(x)\equiv 0$ $(i\leqq n-1)$. One of these is generalised to give a characterisation of harmonic functions by an integral of a second difference. *A. J. Ward* (Cambridge, England).

Bing, R. H. Converse linearity conditions. Amer. J. Math. 68, 309–318 (1946). [MF 16428]

The author gives several sufficient conditions for a continuous function to be linear on a closed interval in terms of points of its graph lying on chords. He thus generalizes previous results of E. F. Beckenbach [Bull. Amer. Math. Soc. 51, 923–930 (1945); these Rev. 7, 246] in terms of midpoints of a chord. *P. Franklin* (Cambridge, Mass.).

Theory of Sets, Theory of Functions of Real Variables

★Fraenkel, Adolf. Einleitung in die Mengenlehre. Dover Publications, New York, N. Y., 1946. viii+424 pp. \$4.00.

Photographic reproduction of the third edition [Springer, Berlin, 1928].

★Hausdorff, F. Mengenlehre. Dover Publications, New York, N. Y., 1944. 307 pp. \$3.50.

Photographic reproduction of the third edition [de Gruyter, Berlin-Leipzig, 1935].

Stone, M. H. On characteristic functions of families of sets. Fund. Math. 33, 27–33 (1945).

This paper, which was to have appeared in 1939, was reprinted in Duke Math. J. 7, 453–457 (1940); cf. these Rev. 2, 256.

Denjoy, Arnaud. Les ensembles rangés. C. R. Acad. Sci. Paris 222, 981–983 (1946). [MF 16387]

The author wishes to discern between the two notions of ordinal numbers (indicating rank) and of types of well-ordered sets, which were identified by G. Cantor. [Cf. the author's note, same C. R. 213, 430–433 (1941); these Rev. 5, 113.] The rank of an element a of an ordered set E has to satisfy the three following conditions: (A) to be invariant under transformations of E into similar sets; (B) to depend only on the (beginning) segment of E; (C) to be characteristic for a (hence, if rank is applicable to the elements of an ordered set E, then any two segments of E have to be dissimilar). A simple example of a (not well-ordered) "ranked set" is given. If a family Γ_0 of ordered sets satisfies certain conditions, a gauging set E_0 of ranks is attached to Γ_0, such that (1) every set E of Γ_0 is similar to a segment of E_0 and (2) every segment of E_0 belongs to Γ_0. Then the element ρ of E_0 having the same rank in E_0 as a has in E is the indicator of the rank. In this way the author generalizes the ordinal numbers. In order to generalize also the types of well-ordered sets, he adjoins to E_0 the set $H(E_0)$ of the types of the segments of E_0 (E_0 itself being excluded). In contradistinction to the types of well-ordered sets, $H(E_0)$ is not similar to E_0 in general. The author indicates the most general type of the ranked sets E_0 which are similar to $H(E_0)$. A more detailed discussion will be given in the author's forthcoming book "L'Enumération Transfini." *A. Rosenthal* (Albuquerque, N. M.).

Szpilrajn-Marczewski, Edward. Sur deux propriétés des classes d'ensembles. Fund. Math. 33, 303–307 (1945). [MF 16895]

A class K of sets has the property (s) if every subfamily of disjoint sets of K is at most countable. A class K has the property (k) if every uncountable subfamily of K contains an uncountable subfamily every two elements of which intersect. The following theorem is proved. A class P has the property (k) if and only if, for every class Q having the property (s), the family of Cartesian products $A\times B$, $A\epsilon P$, $B\epsilon Q$, has the property (s). *S. Eilenberg.*

Kurepa, Georges. Le problème de Souslin et les espaces abstraits. Revista Ci., Lima 47, 457–488 (1945). [MF 15431]

The results established here have been stated earlier [C. R. Acad. Sci. Paris 204, 325–327 (1937); see also the author's thesis, Publ. Math. Univ. Belgrade 4, 1–138 (1935)]. For more recent related work see B. Knaster [Rec. Math. [Mat. Sbornik] 16(58), 281–290 (1945); these Rev. 7, 277] and the paper of W. Sierpiński reviewed below. Let G be the class of open subsets of a space and let Γ be that of all their components. Let $|S|$ be the cardinal of a well-ordered series S of expanding subclasses of a class M; denote by p_cM the upper bound of all $|S|$. Similarly define p_dM and p_sM corresponding to contracting series and to series no two terms of which have elements in common. It is shown that for any infinite locally connected (Fréchet) V-space we have $p_cG=p_c\Gamma+p_s\Gamma$ and $p_dG\geqq p_d\Gamma+p_s\Gamma$. The proofs employ the concept of ordered systems of classes introduced in the author's thesis. Combining the first of these equalities with the fact [Hahn-Mazurkiewicz] $\Gamma\subset G$ it is observed that $p_c\Gamma\leqq p_sG$ implies $p_cG=p_sG$. This implication, in conjunction with known results, leads to the result that, in the case of an infinite locally connected bicompact Hausdorff space satisfying $p_c\Gamma\leqq p_sG$ (and indeed for any ordered connected space), the following conditions are equivalent: (1) the "condition of Souslin" $p_sG\leqq\aleph_0$; (2) $p_cG\leqq\aleph_0$; (3) that any scattered subset is at most countable; (4) that any more than countable subset contains a point of condensation; (5) that any set has the Lindelöf property, i.e. if covered by any system it is covered by a countable subsystem; (6) that every open set is an F_σ; (7) any closed subset is the set of zeros of a function continuous in the whole space. There are stated, in addition, many problems which are equivalent to or closely related with the problem of Souslin. *J. Todd* (London).

Sierpiński, Wacław. Sur un problème de la théorie générale des ensembles. Fund. Math. 33, 299–302 (1945).

If $\mathfrak{A}$ and $\mathfrak{B}$ are two families of sets, let $\mathfrak{A}\dot{\times}\mathfrak{B}$ designate the family of all Cartesian products $A\times B$, where $A\epsilon\mathfrak{A}$ and $B\epsilon\mathfrak{B}$. A family of sets $\mathfrak{X}$ enjoys the property of Souslin if $\mathfrak{X}$ contains no nondenumerably infinite subfamily of nonvoid pair-wise disjoint sets. The family $\mathfrak{X}$ enjoys the property S_1 if it contains no infinite subfamily of nonvoid, pair-wise disjoint subsets. The author shows that, if $\mathfrak{A}$ and $\mathfrak{B}$ both enjoy the Souslin property, $\mathfrak{A}\dot{\times}\mathfrak{B}$ may fail to enjoy that property but that, if $\mathfrak{A}$ and $\mathfrak{B}$ enjoy the property S_1, then $\mathfrak{A}\dot{\times}\mathfrak{B}$ enjoys the property S_1. *E. Hewitt.*

Sierpiński, Wacław. Sur un espace métrique séparable universel. Fund. Math. 33, 115–122 (1945). [MF 16873]

Sierpiński, Wacław. Sur les espaces métriques universels. Fund. Math. 33, 123–136 (1945). [MF 16874]

Detailed proofs of results announced in Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. 75, 571–574, 575–577 (1940) [these Rev. 3, 73].

Licheri, Augusto. Una questione elementare sulla teoria degli insiemi. Rend. Sem. Fac. Sci. Univ. Cagliari 10, 121–122 (1940). [MF 16217]

Proof of the fact that every closed set is the derivative of a set. *A. Rosenthal* (Albuquerque, N. M.).

Maximoff, Isaie. Sur la transformation continue de fonctions. Bull. Soc. Phys.-Math. Kazan (3) 12, 9–41 (1940). (Russian. French summary) [MF 14319]

The purpose of this paper is to prove the following theorem. Let $f(x)$ be a finite function defined on the closed interval $[0, 1]$, belonging to the first class of Baire, and having the property of Darboux on the interval. Then there exists an increasing continuous function $p(t)$ with $p(0)=0$, $p(1)=1$, such that the function $f(p(t))$ is approximately continuous on the interval $0\leqq t\leqq 1$. *J. V. Wehausen.*

Maximoff, Isaie. Sur les fonctions de classe 1 ayant la propriété de Darboux. Bull. Soc. Phys.-Math. Kazan (3) 12, 43–55 (1940). (Russian. French summary) [MF 14322]

The author has previously given a necessary and sufficient condition that a function belonging to the first class of Baire on an interval has the property of Darboux [Prace Mat.-Fiz. 43, 260–265 (1936)]. This condition is in terms of local behavior of the function. In the present paper he gives a condition in terms of global behavior of the function which is, however, too elaborate to give here. The theorem is applied to give a necessary and sufficient condition for the existence of a function $p(x)$ of the first Baire class and having the property of Darboux which satisfies the functional equation $g(x)=f(p(x))$, where $g(x)$ and $f(y)$ are Borel measurable. *J. V. Wehausen* (Washington, D. C.).

Maximoff, Isaie. Sur la transformation continue de quelques fonctions en dérivées exactes. Bull. Soc. Phys.-Math. Kazan (3) 12, 57–81 (1940). (Russian. French summary) [MF 14323]

The author proves the following theorem as the answer to a problem proposed to him by N. Lusin. For every finite function $f(x)$ belonging to the first Baire class and having the property of Darboux on the interval $0\leqq x\leqq 1$, there exists a continuous increasing function $p(t)$, $0\leqq t\leqq 1$, $p(0)=0$, $p(1)=1$, such that $f(p(t))$ is the derivative at every point of $0\leqq t\leqq 1$ of a function $F(t)$. *J. V. Wehausen.*

Frola, Eugenio. Un teorema sulla derivazione delle successioni di funzioni additive. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. 78, 120–124 (1943). [MF 16243]

The author proves the following theorem. Let $\{\varphi_n(\delta)\}$ be a sequence of [totally] additive and absolutely continuous set functions defined in a [σ-] field K of sets and converging there to a [totally] additive and absolutely continuous set function $\varphi(\delta)$; moreover, let $f_n(P)$ and $f(P)$ be the derivatives of $\varphi_n(\delta)$ and $\varphi(\delta)$, respectively. Then $f_n(P)$ converges to $f(P)$ in a closed [and compact] set γ_0 of K, provided the derivatives $f_n(P)$ and $f(P)$ are equicontinuous in γ_0. The proof is given for an n-dimensional Euclidean space but the author states that the proof can be generalized for abstract, separable spaces. (The words in [ ] are added by the reviewer.) *A. Rosenthal* (Albuquerque, N. M.).

Barral Souto, José. Theorems analogous to Rolle's theorem and the law of the mean for continuous functions, based on finite divided differences. Publ. Inst. Mat. Univ. Nac. Litoral 6, 111–120 (1946). (Spanish. English summary) [MF 16914]

Proof that the graph of a continuous function, taking equal values at a and b, has horizontal chords of lengths $(b-a)/n$ if $n=1, 2, \cdots$, but not necessarily if n is not an integer. [The author was evidently not aware that this was proved by P. Lévy, C. R. Acad. Sci. Paris 198, 424–425 (1934); generalizations were given by H. Hopf, Comment. Math. Helv. 9, 303–319 (1937).] *R. P. Boas, Jr.*

Visser, C. An elementary inequality. Nederl. Akad. Wetensch., Proc. 48, 272–275 = Indagationes Math. 7, 77–80 (1945). [MF 15799]

Given s distinct n-letter "words," let s_i $(i=1, \cdots, n)$ denote the number of distinct $(n-1)$-letter words obtained when the ith letter of each given word is deleted. The author shows that $s_1s_2 \cdots s_n \geqq s^{n-1}$ and generalizes this inequality as follows. Let e be the measure of a bounded set E of points $(x_1, \cdots, x_n)$ in n-dimensional space and let e_i be the $(n-1)$-dimensional measure of its projection onto the coordinate space $x_i=0$. (It is assumed for simplicity that E is either open or closed.) Then $e_1e_2 \cdots e_n \geqq e^{n-1}$ and there is equality if and only if E coincides almost everywhere with the direct product of n one-dimensional sets on the n coordinate axes. For $n=3$ it follows that $e_1+e_2+e_3 \geqq 3e^{2/3}$, an inequality established for convex sets by Minkowski [Math. Ann. 57, 447–495 (1903)]. *H. P. Mulholland.*

Pauc, Christian. Construction de mesures. C. R. Acad. Sci. Paris 222, 123–125 (1946). [MF 15984]

A general Boolean algebra is represented as an algebra of subsets of a product of two-point spaces and this representation is used to construct finitely additive measures. *W. Ambrose* (Ann Arbor, Mich.).

Sparre Andersen, Erik, and Jessen, Børge. Some limit theorems on integrals in an abstract set. Danske Vid. Selsk. Math.-Fys. Medd. 22, no. 14, 29 pp. (1946). [MF 16371]

The authors prove two limit theorems on integrals in an abstract set. They can be summarized as follows, omitting some of the conclusions. (A) Let E be a set containing at least one element and let μ be a measure defined on a Borel field $\mathfrak{F}$, with $E\epsilon\mathfrak{F}$ and $\mu(E)=1$. Let $\mathfrak{F}_1 \subseteq \mathfrak{F}_2 \subseteq \cdots$ be an increasing sequence of Borel fields contained in $\mathfrak{F}$, such that $E\epsilon\mathfrak{F}_1$. Let φ be a bounded completely additive set function defined on $\mathfrak{F}$ which is absolutely continuous on each $\mathfrak{F}_n$, so that φ is the integral on $\mathfrak{F}_n$ of an $\mathfrak{F}_n$ measurable function f_n. Then $\lim_{n\to\infty} f_n$ exists almost everywhere on E. (B) Let E, μ and $\mathfrak{F}$ be as in (A) and let $\mathfrak{F}_1 \supseteq \mathfrak{F}_2 \supseteq \cdots$ be a decreasing sequence of Borel subfields of $\mathfrak{F}$, such that $E\epsilon\mathfrak{F}_n$ for every n. Let f be a μ integrable function, with indefinite integral φ. Then the set function φ is absolutely continuous on $\mathfrak{F}_n$ and is therefore the integral on $\mathfrak{F}_n$ of an $\mathfrak{F}_n$ measurable function f_n; similarly it is the integral on $\mathfrak{F}' = \prod_1^\infty \mathfrak{F}_n$ of an $\mathfrak{F}'$ measurable function f'. The theorem now states that $\lim_{n\to\infty} f_n = f'$ almost everywhere on E. [These theorems are essentially the same as two probability theorems of the reviewer, which were evidently unknown to the authors; cf. Trans. Amer. Math. Soc. 47, 455–486 (1940), theorems 1.2 and 1.3; these Rev. 1, 343.] The theorems are applied to show the existence of net derivatives of a set function, the convergence of certain Riemann sums toward the Lebesgue integral of a function, etc. *J. L. Doob* (Urbana, Ill.).

Tonelli, Leonida. Sull'integrazione delle funzioni. Ann. Scuola Norm. Super. Pisa (2) 11, 235–240 (1942). [MF 16764]

Remarks of a pedagogical and expository character on the introduction of the concept of the definite integral. *T. Radó* (Columbus, Ohio).

Tonelli, Leonida. Su alcuni concetti dell'analisi moderna. Ann. Scuola Norm. Super. Pisa (2) 11, 107–118 (1942). [MF 16756]

This is an expository paper, concerned with the concepts of bounded variation and absolute continuity for functions of one variable and functions of several variables, with particular emphasis upon the work of Italian mathematicians. *T. Radó* (Columbus, Ohio).

Mambriani, A. Alcuni legami fra funzioni di due variabili a variazione limitata e funzioni di due variabili a variazione doppia limitata. Boll. Un. Mat. Ital. (2) 5, 150–156 (1943). [MF 16101]

Let $f(x, y)$ be a continuous function of (x, y) in the square $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$; write

$$F_{(y)}(x, y) = \int_0^x f(u, y)du, \quad V_{(y)}(x) = \int_0^1 |d_v f(x, v)|,$$

and similarly for $F_{(x)}(x, y)$ and $V_{(x)}(y)$. The author observes that $f(x, y)$ is of bounded variation in Tonelli's sense if and only if $F_{(y)}(x, y)$ and $F_{(x)}(x, y)$ are of bounded (double) variation in the sense of Vitali; he goes on to show that the total double variation of $F_{(y)}(x, y)$ over the square equals the integral over (0, 1) of $V_{(y)}(x)$, and similarly for $F_{(x)}(x, y)$. The integrals just mentioned are, of course, those whose finiteness is required by Tonelli's definition. *H. P. Mulholland* (Beirut).

Ridder, J. Denjoy-Stieltjessche und Perron-Stieltjessche Integration im k-dim. Euklidischen Raum. Nieuw Arch. Wiskunde (2) 21, 212–241 (1943). [MF 15705]

The function $\Phi(E)$ is completely additive on the bounded Borel sets of a k-dimensional space R_k. For such a function there are positive, negative and total variations, P, N and T, with $\Phi(E)=P(E)-N(E)$, $T(E)=P(E)+N(E)$. Sequences of sets $A_n(X)$ associated with the points X of R_k are defined as follows. For $\alpha>1$, $r_1> \cdots >r_n \to 0$, $K(X, r_n)$, $K(X, \alpha r_n)$ spheres with center X, then $K(X, r_n) \subseteq A_n(X) \subseteq K(X, \alpha r_n)$. If Φ and Ψ are completely additive, and $\Phi[A_{n_j}(X)] \neq 0$ for an infinite set of values n_j of n,

$$mA_{n_j}(X) = \Psi[A_{n_j}(X)]/\Phi[A_{n_j}(X)],$$

then $\bar{m}A_n(X) = \limsup mA_{n_j}(X)$; $\underline{m}A_n(X) = \liminf mA_{n_j}(X)$. Let $\bar{D}_\Phi\Psi(X)$ be the lower bound of all $\bar{m}A_n(X)$, $\underline{D}_\Phi\Psi(X)$ the upper bound of all $\underline{m}A_n(X)$. If these two bounds are equal their common value is $D_\Phi\Psi(X)$. If Φ is completely additive then $D_T\Phi=1$ or -1 except for a set A with $\bar{m}_T(A)=0$. If Ψ is completely additive $D_\Phi\Psi$ exists except for A with $\bar{m}_T(A)=0$. If Ψ is the indefinite Lebesgue-Stieltjes integral with respect to Φ of a function $f(X)$ then $D_\Phi\Psi=f$ except for a set A with $\bar{m}_T(A)=0$.

A class of elements called a Burkill-class is defined. If E is a set in this class, $\int_E F$ is a Burkill integral. It is shown that, for a completely additive function Φ, $D_\Phi F = D_\Phi \int_E F \neq \pm\infty$, except for a set A with $\bar{m}_T A=0$. A function F defined on a bounded closed region of elements of a B-class is absolutely continuous with respect to a nonnegative additive function H of elements of a B-class if for $\epsilon>0$ there exists $\eta>0$ such that, for mutually exclusive sets $A_1, \cdots, A_l$ with $\sum H(A_j) < \eta$,

$\sum F(A_j)<\epsilon$. Based on this definition of absolute continuity, upper and lower semi-absolute continuity and generalized absolute continuity with respect to H are defined and these concepts used to arrive at a definition of a Denjoy-Stieltjes integral in a manner similar to that used in the case of functions of a single real variable. Using derivatives with respect to H, major and minor functions to a function $f(x)$ are defined, and these lead to a Perron-Stieltjes integral with respect to H. It is shown that these Denjoy-Stieltjes and Perron-Stieltjes integrals are equivalent.

R. L. Jeffery (Kingston, Ont.).

Federer, Herbert. Coincidence functions and their integrals. Trans. Amer. Math. Soc. 59, 441–466 (1946). [MF 16467]

The author treats principally three problems. First, he establishes a relation, known previously in special cases, between the integral of the coincidence function of a k-dimensional surface and an $(n-k)$-dimensional surface in Euclidean n-space E_n on the one hand and the product of the areas of the surfaces on the other. Let f be a function defined on a measurable subset U of E_k with values in E_n, and let g be defined on a measurable subset V of E_{n-k} with values in E_n. Define $\Omega[f, U; g, V]$ to be the number (possibly ∞) of ordered pairs (u, v) for which $u\,\epsilon\,U$, $v\,\epsilon\,V$ and $f(u)=g(v)$. Hold f fixed and apply a distance preserving transformation $R:T_z$ to the surface g. Then the coincidence function of f and g is $\Omega[f, U; (R:T_z:g), V]$. It is defined for each $R:T_z$ in the group K_n of all distance preserving transformations of E_n. Let G_n denote the group of orthogonal transformations R of E_n, and let $T_z\,\epsilon\,K_n$ denote the translation of E_n associated with $z\,\epsilon\,E_n$ by the relation $T_z(x)=z+x$ for $x\,\epsilon\,E_n$. Metrize K_n appropriately so that G_n is a compact topological group, and introduce the unique Haar measure φ_n in G_n such that $\varphi_n(G_n)=1$. Then it is possible to consider integrals

$$\int_{G_n}\int_{E_n} h(R:T_z)\,d\mathfrak{L}_n^n z\,d\varphi_n R$$

for functions h defined on K_n to E_1. Here $\mathfrak{L}_n^n$ denotes n-dimensional Lebesgue measure. As the principal result for the first problem, the author proves the following theorem. If f is countably Lipschitzian on the $\mathfrak{L}_n^n$ measurable subset U of E_k to E_n, and if g is countably Lipschitzian on the $\mathfrak{L}_{n-k}^{n-k}$ measurable subset V of E_{n-k} to E_n, then

$$(1)\quad \int_{G_n}\int_{E_n} \Omega\{f, U; (R:T_z:g), V\}\,d\mathfrak{L}_n^n z\,d\varphi_n R \\ =\beta(n, k)\cdot\int_U Jf(x)\,d\mathfrak{L}_k^k x\cdot\int_V Jg(y)\,d\mathfrak{L}_{n-k}^{n-k} y.$$

The integrals on the right are the areas of the two surfaces. The constant $\beta(n, k)$ depends only on the dimensions n and k and the author determines its value explicitly in terms of the gamma function.

Poincaré proved [see his Calcul des Probabilités, 2d ed., Gauthier-Villars, Paris, 1912] the equality (1) for the case in which f and g are two curves in the plane (the special case $n=2$, $k=n-k=1$). For a curve and a line segment it had previously been obtained by Crofton. The integral geometry of Blaschke and his students [see W. Blaschke, Vorlesungen über Integralgeometrie, Hamburger Math. Einzelschr. 20, 22, Teubner, Leipzig, 1936, 1937] is related to the problem. Santaló obtained [see his Integralgeometrie 5. Über das kinematische Mass im Raum, Actual. Sci. Ind., no. 357, Hermann, Paris, 1936] the result (1) for a curve and an ordinary surface in 3-space (the special case $n=3$, $k=1$, $n-k=2$). In the paper under review the author establishes the general result (1) for the first time.

The second of the three problems is concerned with a new definition of area. The author defines an area for all continuous k-dimensional surfaces in terms of the stable values [for a treatment of stable values see p. 74 of W. Hurewicz and H. Wallman, Dimension Theory, Princeton Mathematical Series, vol. 4, 1941; these Rev. 3, 312] of their projections into k-dimensional subspaces; the area thus defined is lower semi-continuous. The author claims that this area has technical advantages over those of Peano and Geöcze. Its relation to the Lebesgue area is only partially settled in this paper.

The third main result of the paper grew out of certain problems raised by the paper itself. The author proves that the Gauss-Green theorem holds for every bounded open subset of Euclidean n-space whose boundary has finite $(n-1)$-dimensional Hausdorff measure. This theorem supplements the results of an earlier paper by the author [Trans. Amer. Math. Soc. 58, 44–76 (1945); these Rev. 7, 199].

The paper as a whole is characterized by the treatment of problems and the employment of methods of great generality. The author uses many results from two of his previous papers [Trans. Amer. Math. Soc. 55, 420–437, 438–456 (1944); these Rev. 6, 44, 45]; in addition, he employs a wide variety of powerful tools selected freely from the theory of topological groups, measure theory, integration theory, the theory of functions of real variables, topology and other fields of modern mathematics.

G. B. Price (Lawrence, Kan.).

Morse, Anthony P., and Randolph, John F. Gillespie measure. Fund. Math. 33, 12–26 (1945).

This paper, which was to have appeared in 1939, was reprinted in Duke Math. J. 6, 408–419 (1940); cf. these Rev. 1, 304.

Busemann, Herbert. Intrinsic area. Proc. Nat. Acad. Sci. U. S. A. 32, 5–8 (1946). [MF 15310]

The author defines an n-dimensional area for surfaces S defined by uniformly continuous parametric representations $S:x=x(p)$, where $p\,\epsilon\,P$, $x\,\epsilon\,R$, P and R being metric spaces, P being also separable with P^* arcwise connected (P^* is the smallest complete metric space containing P). In order to do this, the writer defines $\kappa[q, r, x(P)]$ for points q and r in P^* as the greatest lower bound of the lengths of the curves $x=x[p(\tau)]$, where $p=p(\tau)$ is a continuous curve in P^* which joins q and r. New spaces $\bar{P}$ and $\bar{P}^*$ are formed by identifying pairs of points q and r for which $\kappa[q, r, x(P)]=0$ and then using this function as a metric. The area $\alpha_n(S)$ is then defined as the n-dimensional Hausdorff measure of $\bar{P}$, considered as a subset of itself. This area satisfies the following conditions (and a few other similar ones): (1) $\alpha_n(S)$ is independent of the representation; (2) $\alpha_1(S)$ is the arc length if S is a curve; (3) if $\alpha_n(S)>0$, $\alpha_i(S)=+\infty$ if $i<n$; if $\alpha_n(S)<\infty$, $\alpha_i(S)=0$ for $i>n$; (4) if S is a rectifiable surface (that is, $x(p)$ is Lipschitzian for some representation of S) in E_n and P is the unit cube in E_k, then $\alpha_k(S)$ coincides with the Lebesgue area; (5) $\alpha_n(S)$ depends only on the intrinsic geometry of S; (6) if $T:y=y(P)$ is another mapping on P into another space R' in which $y(q)y(r)\leqq\beta\cdot x(q)x(r)$, then $\alpha_n(T)\leqq\beta^n\alpha_n(S)$. Moreover, in Riemannian spaces, $\alpha_n(S)$ is the only area which satisfies (4), (5) and (6) for surfaces of class C'.

C. B. Morrey, Jr. (Berkeley, Calif.).

Bouligand, Georges. Recherche opératoire de courbes et surfaces rectifiables. C. R. Acad. Sci. Paris **222**, 120–122 (1946). [MF 15983]

Let x, y be rectangular coordinates in the Euclidean plane. Let $f(x)$ be a continuous monotonically increasing function whose difference quotients are greater than 1 and bounded (thus $f(x)$ has a derivative almost everywhere). Then the most general rectifiable curve $y=y(x)$ through the origin, the arc length of which between that point and the point $(x, y(x))$ is equal to $f(x)$, is given by the Lebesgue integral

$$y(x)=\int_0^x \epsilon(v)(f'^2(v)-1)^{\frac{1}{2}}dv,$$

where the measurable function $\epsilon(v)$ is equal to ±1 almost everywhere.

Similar problems are discussed for Lebesgue-rectifiable surfaces with given ds. Their solution involves one or two such functions ϵ. One example may suffice. Let $\mathfrak{e}(v)$ be the vector representation of a rectifiable curve on the unit sphere. Let A, B, C, D be given functions of v, continuous and with bounded difference quotients such that the form

$$ds^2=du^2+(Au^2+2Bu+C)dv^2+2Ddudv$$

is positively definite $(A=\mathfrak{e}'^2)$. If $\mathfrak{y}(v)$ is to be a rectifiable curve in 3-space such that the surface $\mathfrak{y}(v)+u\mathfrak{e}(v)$ has the above ds^2, then $A\mathfrak{y}'=AD\mathfrak{e}+B\mathfrak{e}'+\epsilon(v)(AC-B^2-AD^2)^{\frac{1}{2}}(\mathfrak{e}\times\mathfrak{e}')$ almost everywhere. *P. Scherk* (Saskatoon, Sask.).

Okamura, Hirosi. Sur une sorte de distance relative à un système différentiel. Proc. Phys.-Math. Soc. Japan (3) **25**, 514–523 (1943). [MF 15066]

If B is a closed and bounded set in $(x, y)=(x, y_1, \cdots, y_n)$ space whose projection on the x-axis is the interval $[\alpha, \beta]$ and such that there exists a curve $y=g(x)$ with $(x, g(x))$ in B, $\alpha\leqq x\leqq\beta$, and $g(x)$ of bounded variation in $[\alpha, \beta]$, while $F(x, y)$ is a given vector function continuous in B, then the distance $D(P, Q)$ between two points $P=(x_P, y_P)$ and $Q=(x_Q, y_Q)$ of B is defined as follows: if $x_P\neq x_Q$, $D(P, Q)$ is the greatest lower bound of the total variation of $y(x)-\int F(x, y(x))dx$ on $[x_P, x_Q]$ in the class of vector functions $y=y(x)$ which are of bounded variation on this interval and satisfy $y(x_P)=y_P$, $y(x_Q)=y_Q$, $(x, y(x))$ in B for x on $[x_P, x_Q]$; if $x_P=x_Q$, then $D(P, Q)=|y_P-y_Q|$. The author discusses properties of this distance function, which clearly depends upon the set B and the function $F(x, y)$, and then applies it to the proof of a general criterion for the uniqueness of solutions of the vector differential equation $dy/dx=F(x, y(x))$. *W. T. Reid* (Evanston, Ill.).

Theory of Functions of Complex Variables

***Botella Raduan, Francisco. Los Espacios de Riemann y la Teoria de Funciones. [Riemann Spaces and the Theory of Functions].** Consejo Superior de Investigaciones Cientificas, Madrid, 1942. 72 pp. (Spanish)

Relations between the theory of two-dimensional Riemannian manifolds (in particular, those of constant curvature) and the theory of analytic functions of a single complex variable are studied. *M. H. Heins* (Providence, R. I.).

Perelmann, M. Sur le module de continuité des fonctions analytiques. Leningrad State Univ. Annals [Uchenye Zapiski] **83** [Math. Ser. **12**], 62–86 (1941). (Russian. French summary) [MF 16489]

Let $\psi(\delta)$ be the modulus of continuity of the function $f(x)$ defined and analytic on a closed interval (a, b) (thus $\psi(\delta)=\sup|f(x_2)-f(x_1)|$ for $a\leqq x_1<x_2\leqq b$, $x_2-x_1\leqq\delta$). The function $\psi(\delta)$ is analytic in a sufficiently small interval $(0, \epsilon)$, depending on f. *A. Zygmund* (Philadelphia, Pa.).

Wall, H. S. Continued fraction expansions for functions with positive real parts. Bull. Amer. Math. Soc. **52**, 138–143 (1946). [MF 15454]

The author proves the following theorem. Let R denote the complex z-plane with a cut along the real axis from -1 to $-\infty$. If $c>0$, $0<g_p<1$, $-\infty<r_{p-1}<\infty$, $p=1, 2, 3, \cdots$, the continued fraction

$$(1)\qquad c(1+z)^{\frac{1}{2}}/\{t_0+g_1z/[t_1+K_1(1-g_n)g_{n+1}z/t_{n+1}]\},$$

where $t_n=1+ir_n(1+z)^{\frac{1}{2}}$, converges uniformly in every bounded region whose closure lies in R to a function $F(z)$ which is analytic and has a positive real part throughout R. Conversely, if $F(z)$ is any function having these properties there exists a uniquely determined continued fraction of the form (1) whose value is $F(z)$. *W. Leighton.*

Ríos, Sixto. Sur l'ultraconvergence des séries d'interpolation. C. R. Acad. Sci. Paris **222**, 168–169 (1946). [MF 15989]

Extending a well-known theorem of Landau, the author proves that, if the partial sums corresponding to the indices $\{n_\nu\}$ of one of the three series

$$\sum a_n n^{-s},\quad \sum n!a_n\{s(s+1)\cdots(s+n)\}^{-1},$$
$$\sum a_n(s-1)(s-2)\cdots(s-n)(n!)^{-1}$$

converges uniformly in a region D around a regular point of its axis of convergence (distinct from 0, $\pm n$), then this is also true for the other two series. *S. Mandelbrojt.*

Traupel, W. Calculation of potential flow through blade grids. Sulzer Tech. Rev. **1945**, no. 1, 25–42 (1945). [MF 14077]

The author gives a method for calculating the two-dimensional flow through an arbitrarily given cascade of airfoils, with the inflow direction making an arbitrary angle with the cascade axis. The essential step is to apply the transformation $\zeta=(e^z+1)/(e^z-1)$, which maps the exterior of a cascade with gap 2π in the z-plane into an infinitely many sheeted Riemann surface bounded by a single closed curve in the ζ-plane, having branch points at $\zeta=\pm1$. The problem is then reduced to the solution of a potential problem in a simply connected region, where the potential has logarithmic singularities at $\zeta=\pm1$. Further conformal transformations are used to simplify the problem. The method of successive approximations is then used for actually carrying out the calculations. Practical procedure and a numerical example are given. *C. C. Lin.*

Komatu, Yûsaku. Über das Randverhalten beschränkter Schlitzabbildungen und seine Anwendungen. Proc. Phys. Math. Soc. Japan (3) **24**, 187–197 (1942). [MF 15024]

Application is made of Löwner's differential equation associated with bounded "Schlitzabbildungen" [Math. Ann. **89**, 103–121 (1923)] to the determination of the angular derivative of a bounded Schlitzabbildung and to the work of V. Paaterö [Ann. Acad. Sci. Fennicae. Ser. A. **48**, no. 10 (1937)] on bounded functions which carry assigned boundary arcs into assigned boundary arcs. *M. H. Heins.*

Bolder, H. Sur quelques propriétés extrémales du domaine de Koebe. Nederl. Akad. Wetensch., Proc. 48, 216–221 = Indagationes Math. 7, 21–26 (1945). [MF 15793]

A number of extremal properties for the Green's function of a Koebe "slit" region are established. These properties correspond to classical theorems on univalent functions.

M. H. Heins (Providence, R. I.).

Calugareanu, Georges. Sur une représentation conforme des domaines multiplement connexes. Bull. Math. Soc. Roumaine Sci. 46, 33–41 (1944). [MF 16508]

Pólya and Szegö proved that the capacity of a bounded continuum Γ in the complex plane has the following interpretation. Let G be the component of the complement of Γ containing the point at infinity; let the function $w=\varphi(z)$, with $\varphi'(\infty)=1$, map G conformally and biuniquely on the exterior of a circle γ whose center is at the origin. Then the radius of the unique circle γ is equal to the capacity C of Γ [J. Reine Angew. Math. 165, 4–49 (1931)].

The main purpose of this note is to prove an analogous result for plane sets Γ^* which consist of at most a finite number of domains. This is done by mapping that component G^* of the complement of Γ^* containing the point at infinity on the exterior of a circle γ^* with center at the origin, by means of the function $w=\Phi(z)=c^*e^{g+ih}$, $\Phi'(\infty)=1$. Here g is the Green's function for G^* and h is its (non-uniform) harmonic conjugate; moreover, the mapping $w=\Phi(z)$ sets up a biunique conformal representation of an infinitely many sheeted Riemannian domain over G^* upon an infinitely many sheeted Riemannian domain over the exterior of the circle γ^* noted above. The capacity of C^* is then shown to be equal to the radius of the circle γ^*.

Other results are proved. For example, the author proves that $\phi(z)=\lim_{n\to\infty}\{|T_n(z)|\}^{1/n}$, where $T_n(z)$ is the nth Chebyshev polynomial for Γ^* and where that branch of the root is taken whose derivative has the value unity for $z=\infty$.

M. O. Reade (Ann Arbor, Mich.).

Hössjer, Gustav. Über die konforme Abbildung eines veränderlichen Bereiches. Trans. Chalmers Univ. Tech. Gothenburg [Chalmers Tekniska Högskolas Handlingar] 1942, no. 10, 15 pp. (1942). [MF 15752]

The author is concerned with the following situation. Let X denote a region in the x-plane and let $Y(x)$ denote a variable simply-connected region in the y-plane which depends upon the parameter $x\varepsilon X$. The boundary of $Y(x)$ is a closed Jordan curve and is denoted by $T(x)$; an inner point of $Y(x)$ is distinguished and is denoted by $Q(x)$. It is required that $Q(x)$ is analytic for $x\varepsilon X$ and that $T(x)$ is given by $q(t,x)$, where q is defined and continuous on the product set $[|t|=1]\cdot[X]$ and for each t is analytic in x. Furthermore, it is required that, for each $x\varepsilon X$ and $t_1\neq t_2$, $q(t_1,x)\neq q(t_2,x)$. Finally, $z(y;x)$ denotes a 1-1 directly conformal mapping of $Y(x)$ onto $|z|<1$ with $z(Q(x);x)=0$. It is shown by potential-theory methods that $\log|z_y(Q(x);x)|$ is subharmonic in x. Some extensions to variable simply-connected Riemannian domains over the y-plane are given. This work is related to earlier studies of the author [Acta Univ. Lundensis [Lunds Univ. Årsskrift] N.S. Sect. 2. 24, no. 9 (1929); 28, no. 11 (1932) = Acta Reg. Soc. Physiog. Lund. [Kungl. Fysiog. Sällskapets i Lund Handlingar] N.S. 39, no. 9; 43, no. 11]. *M. H. Heins* (Providence, R. I.).

Schaeffer, A. C., and Spencer, D. C. The coefficients of schlicht functions. III. Proc. Nat. Acad. Sci. U. S. A. 32, 111–116 (1946). [MF 16364]

[For parts I and II cf. Duke Math. J. 10, 611–635 (1943); 12, 107–125 (1945); these Rev. 5, 175; 6, 206.] Given a point $(a_2, a_3, \cdots, a_n)$, the function

$$f(z)=z+\sum_2^\infty b_\nu z^\nu$$

is said to belong to this point if $f(z)$ is regular and schlicht for $|z|<1$ and if $b_i=a_i$, $i=2,3,\cdots,n$. The region of variability V_n of the coefficients lies in E_{2n-2}. It is known that V_2 consists of the circle $|a_2|\leqq 2$. In this note the authors outline a method which gives V_n for general n. In particular, the boundary of V_3 is expressed by equations involving only elementary functions, but for $n>3$ the boundary of V_n cannot be so expressed and is quite complicated.

Let $F(a_2,\bar a_2,\cdots,a_n,\bar a_n)$ be defined in a region B_n containing V_n. Let F be real, F and its first order derivatives F_ν continuous and $\sum_2^n|F_\nu|^2>0$; F can have its maximum value in V_n only at a boundary point of V_n. If f maximizes F then f satisfies the differential equation

$$(1)\qquad (zf'(z))^2\sum_2^n A_\nu f(z)^{-\nu-1}=B+\sum_{\nu=1}^{n-1}(B_\nu z^{-n+\nu}+\bar B_\nu z^{n-\nu}),$$

where

$$A_\nu=\sum_{k=\nu}^n a_k^{(\nu)}F_k,\quad B_\nu=\sum_{k=1}^\nu ka_kF_{n+k-\nu},\quad B=\sum_{k=2}^n(k-1)a_kF_k,$$

with $(f(z))^\nu=\sum_{k=\nu}^\infty a_k^{(\nu)}z^k$. To every point on the boundary of V_n there belongs at least one f which satisfies an equation of type (1).

A function $f(z)$, $f(0)=0$, $f'(0)=1$, regular in $|z|<1$ is called a D-function if it satisfies an equation of type (1), where the A's and B's are constants such that B is real and $\sum_2^n|A_\nu|^2=1$ and $B+\sum_{\nu=1}^{n-1}(B_\nu z^{-n+\nu}+\bar B_\nu z^{n-\nu})\geqq 0$ on $|z|=1$. There is a one-to-one correspondence between boundary points of V_n and D-functions. Any D-function $f(z)$ is schlicht and $w=f(z)$ maps $|z|<1$ onto the w-plane deleted by the piecewise analytic locus from $w=\infty$ which satisfies the Schiffer differential equation. If an $f(z)$ belonging to a boundary point of V_n satisfies more than one differential equation D, then $f(z)$ is algebraic.

Let $V_3^{(0)}$ be the subset of V_3 for which a_2 is real; $V_3^{(0)}$ is symmetrical with respect to the plane $a_3=0$; V_3 is obtained from $V_3^{(0)}$ by rotations. The portion of $V_3^{(0)}$ for which $a_3\geqq 0$ is completely described by two analytic surfaces and their intersection. If $w=f(z)$ belongs to a point of the first of these surfaces it generally maps $|z|<1$ on the w-plane minus a forked slit composed of the ray amp (w) = constant extending from $w=\infty$ to some finite point where there is a fork of two prongs which form angles $2\pi/3$ with the ray. In special cases one or both prongs may degenerate to a point. Analogous statements are made for the second surface. Detailed proofs are to appear later.

M. S. Robertson (New Brunswick, N. J.).

Onofri, Luigi. Contributo alla teoria delle funzioni univalenti. Boll. Un. Mat. Ital. (2) 4, 217–224 (1942). [MF 16072]

If $f(z)=z+a_2z^2+\cdots$ is holomorphic in $|z|<1$, real for z real, and if, for $0<|z|<1$ and $\Im(z)>0$, $\Im(zf'(z))>0$, then it represents a function univalent in $|z|<1$. If, moreover, $|a_n|<1$ $(n>1)$ the corresponding function maps $|z|<1$ on a region containing the circle $|z|<\frac{1}{2}$. If $0\leqq na_n<M$, the set $\{na_n\}$ being nowhere concave, the function $f(z)=z+a_2z^2+\cdots$ is univalent in $|z|<1$. Other theorems of this kind, involving n^2a_n, are given. *S. Mandelbrojt* (Paris).

Tumura, Yosiro. Sur une extension d'un théorème de M. Teichmüller. Proc. Imp. Acad. Tokyo 19, 55–59 (1943). [MF 14793]

The following theorem generalizing one due to Teichmüller [Deutsche Math. 2, 96–107 (1937)] is established. Let G denote a plane multiply-connected region in the z-plane of connectivity n for which each boundary component is a continuum. Let S denote its transversal lying on a line and satisfying the condition that the subset of segments of S whose extremities lie on distinct boundary components either divides G or is vacuous. Let $g(z, z_0)$ denote the Green's function with respect to G having its pole at z_0. Then there exists a positive constant C independent of S such that

$$(1/2\pi)\int_S g(z, z_0)|dz| \leqq CL,$$

where L is the length of S. Applications are made to a theorem of Collingwood and Teichmüller [loc. cit.].

M. H. Heins (Providence, R. I.).

Spitzbart, A. Approximation in the sense of least pth powers with a single auxiliary condition of interpolation. Bull. Amer. Math. Soc. 52, 338–346 (1946). [MF 16202]

Die Funktion $w=g(z)$ bilde das Innere der Jordan-Kurve C konform auf das Innere des Einheitskreises $|w|=1$ ab. Es wird $f(z)$ als zur Klasse E_p zugehörig bezeichnet, wenn

$$\int_{C_r}|f(z)|^p|dz|$$

beschränkt ist für $r<1$, wo C_r die Kurve $|g(z)|=r$ bedeutet. Der Verfasser beweist neben einigen Verallgemeinerungen den folgenden Satz. Von der Klasse E_p sei insbesondere diejenige eindeutig bestimmte Funktion $F_0(z)$ betrachtet, welche

$$\int_C|f(z)|^p|dz|$$

zum Minimum macht, wobei alle Funktionen dieser Klasse der Normierung genügen sollen $f(\alpha)=A$ (α, A beliebig und fest). Es sei $P_n(z)$ das entsprechende Minimalpolynom für das vorige Integral mit $P_n(\alpha)=A$. Die Folge $P_n(z)$ konvergiert maximal [im Sinne von Walsh, Interpolation and Approximation by Rational Functions in the Complex Domain, Amer. Math. Soc. Colloquium Publ., vol. 20, New York, 1935] gegen $F(z)$ in dem durch C abgeschlossenen Bereich.

W. Saxer (Zürich).

Nassif, M. On the zeros of basic sets of polynomials. Proc. Math. Phys. Soc. Egypt 2 No. 4, 1–6 (194[illegible]). [MF 16831]

Let $\{P_n(z)\}$ be a simple basic set of polynomials [J. M. Whittaker, Interpolatory Function Theory, Cambridge University Press, 1935] and let the zeros of $p_n(z)$ all lie in the circle $|z|\leqq K_n$. If $K_n=K$, the corresponding basic series represents every entire function of increase less than order 1, type $2/(3K)$. If $K_n=an^\alpha$, the series represents every entire function of increase less than order $1/(\alpha+1)$, type $(\alpha+1)\{a(1+\frac{1}{2}e^\alpha)\}^{-1/(\alpha+1)}$.

R. P. Boas, Jr.

Marcouchewitch, A. Sur les bases dans l'espace des fonctions analytiques. Rec. Math. [Mat. Sbornik] N.S. 17(59), 211–252 (1945). (Russian. French summary) [MF 16669]

The author considers the space E_R of functions of the complex variable z, analytic in $|z|<R$, with the "norm"

$$\|f(z)\|=\sum_{n=1}^{\infty}\frac{M(r_n)}{2^n(1+M(r_n))}, \qquad r_n\to R,$$

where $M(r)$ is the maximum of $|f(z)|$ on $|z|=r$; E_R is a space of type F in Banach's terminology. He considers criteria for the following sequence of four properties which may be possessed by a sequence $\{f_n(z)\}$ of elements of E_R. (1) The closed linear manifold determined by $\{f_n\}$ is E_R; that is, $\{f_n\}$ is fundamental ("complete," in the author's terminology). If (1) is true, any $f(z)$ of E_R has representations $f(z)=\lim_{n\to\infty}\sum_{k=0}^{n}a_k^{(n)}f_k(z)$. Property (2) is that $\lim_{n\to\infty}a_k^{(n)}=L_k(f)$ exists for every representation and every k; then $\{f_n\}$ is called strongly linearly independent and $\{L_k\}$ is a system of linear functionals biorthogonal to $\{f_n\}$. If (2) is true, every element of E_R has a development $\sum L_n(f)f_n(z)$. (3) This expansion is unique; then $\{f_n\}$ is called a base in the wide sense. (4) The expansion of every f of E_R converges to f; then $\{f_n\}$ is a base (in the sense of Schauder).

Most of the author's results for (1) were announced in C. R. (Doklady) Acad. Sci. URSS (N.S.) 43, 3–6 (1944) [these Rev. 6, 127]. He gives a general criterion for (1) from which he derives as particular cases extensions of a variety of known results on completeness of sets of the forms $f(z\zeta_n)$, $z^nf^{(n)}(z)$, $z^nf^{(n)}(z\zeta_n)$, $z^{-n}\int_0^z f(\zeta)(z-\zeta)^{n-1}d\zeta$, etc. [cf. also the following review]. For (2), let $f_n(z)=\sum_{j=0}^{\infty}a_n^{(j)}z^j$ and let (1) be true. Let

$$z^j=\lim_{n\to\infty}\{\alpha_{n,0}^{(j)}f_0(z)+\cdots+\alpha_{n,n}^{(j)}f_n(z)\}, \quad j=0, 1, 2, \cdots,$$

where $\lim_{n\to\infty}\alpha_{n,k}^{(j)}=\alpha_k^{(j)}$ for each k and j. A necessary and sufficient condition for (2) is that

$$\limsup_{j\to\infty}|a_k^{(j)}|^{1/j}=\rho_k<R, \quad k=0, 1, 2, \cdots,$$

$$\sum_{j=0}^{\infty}a_n^{(j)}\alpha_m^{(j)}=\delta_{nm}, \qquad n, m=0, 1, 2, \cdots.$$

Conditions for (3) and (4) are given in terms of an associated system $\omega_n(z)$, which always exists if (2) is satisfied, such that $\omega_n(z)$ is analytic for $|z|>l_n$, $l_n<R$, $\omega_n(\infty)=0$ and $\int f_n(z)\omega_k(z)dz=2\pi i\delta_{kn}$, the integration being around a circle of radius less than R.

R. P. Boas, Jr.

Boas, R. P., Jr. Fundamental sets of entire functions. Ann. of Math. (2) 47, 21–32 (1946). [MF 15656]

A set $\{u_n(z)\}$ of analytic functions is called fundamental or complete, if each function analytic in $|z|<1$ can be uniformly approximated by linear combinations of the $u_n(z)$. The author discusses fundamental sets of the form $F(a_nz)$ or $F(z+a_n)$, where $F(z)$ is an entire function. If $u_n=F(z+a_n)$, $F(z)$ is supposed not to be a finite sum of exponentials with polynomial coefficients (hypothesis T), and, if $u_n=F(a_nz)$, to satisfy the conditions $F^{(n)}(0)\neq0$, $n=0, 1, 2, \cdots$ (hypothesis P). A sequence $\{a_n\}$ is called a uniqueness set of class (ρ, τ) if $f(a_n)=0$, $n=1, 2, \cdots$, implies $f(z)\equiv0$ for every entire function of class (ρ, τ), that is, of order ρ and type less than τ. The following theorem reduces the problem of fundamental sets $\{F(a_nz)\}$ [$\{F(z+a_n)\}$] to the problem of uniqueness sets $\{a_n\}$. If $\{a_n\}$ is a uniqueness set of class (ρ, τ) [$(1, \tau)$], then $\{F(a_nz)\}$ [$\{F(z+a_n)\}$] is fundamental for each function $F(z)$ of class (ρ, τ) [$(1, \tau)$] which satisfies the hypothesis P [T]. Conversely, if $\{a_n\}$ is not a uniqueness set of class (ρ, τ) [$(1, \tau)$], there exist functions $E(z)$ of class (ρ, τ_1) [$(1, \tau_1)$], $\tau_1>\tau$, satisfying hypothesis P [T] with $\{E(a_nz)\}$ [$\{E(z+a_n)\}$] not fundamental. The hypotheses P and T are essential. Many sufficient conditions are found for sets to be uniqueness sets of class (ρ, τ) and hence for sets $\{F(a_nz)\}$ and $\{F(z+a_n)\}$ to be

fundamental. The detailed theorems generalize results of N. Levinson [Gap and Density Theorems, Amer. Math. Soc. Colloquium Publ., vol. 26, New York, 1940; these Rev. **2**, 180], A. Gelfond [Rec. Math. [Mat. Sbornik] N.S. **4**(46), 149–156 (1938)] and A. Marcouchewitch [see the preceding review]. *A. Pfluger* (Zurich).

Boas, R. P., Jr. Functions of exponential type. V. Duke Math. J. **12**, 561–567 (1945). [MF 15501]

Der Verfasser untersucht ganze Funktionen vom Exponentialtypus, deren Nullstellenfolge $\{\lambda_n\}_{n=-\infty}^{\infty}$ messbar ist und nur wenig von der symmetrischen Verteilung ($\lambda_{-n}=-\lambda_n$) abweicht. Es sei

$$f(z)=\prod_1^\infty(1-z/\lambda_n)(1-z/\lambda_{-n}),$$

$|\lambda_{n+1}-\lambda_n|>\delta>0$, $|\lambda_n+\lambda_{-n}|=O(n^\beta)$, $\beta<1$, und $n(r;\theta_1,\theta_2)=(N(\theta_2)-N(\theta_1))r+o(r)$. Dann besitzt der Indikator der ganzen Funktion $f(z)$ die Darstellung

$$h(\varphi)=\pi\int^{\pi}\sin\theta\cdot dN_\theta(\varphi+\theta).$$

Beweis durch Zurückführung auf den Fall symmetrischer Verteilung bzw. ganzer Funktionen der Ordnung $\frac{1}{2}$ und damit auf ein Resultat des Referents [Comment. Math. Helv. **11**, 180–214 (1938)]. [Für Resultate ähnlicher Art, vgl. folgendes Referat.] *A. Pfluger* (Zürich).

Pfluger, A. Über ganze Funktionen ganzer Ordnung. Comment. Math. Helv. **18**, 177–203 (1946).

In earlier papers [same Comment. **11**, 180–214 (1938); **12**, 25–65 (1939); these Rev. **1**, 113] the author discussed the influence of the distribution of the zeros of an entire function of nonintegral order on its growth. Here he considers the more complicated situation for functions of integral order ρ. The zeros of $f(z)$ are assumed measurable, in the sense that the number in $|z|\leqq r$, $\varphi'\leqq\arg z<\varphi''$ is asymptotically $\{N(\varphi'')-N(\varphi')\}r^\rho$ for all points of continuity of a nondecreasing function $N(\varphi)$. Let $z^me^{P(z)}\pi(z)$ be the Hadamard factorization of $f(z)$, let c_ρ be the coefficient of z^ρ in $P(z)$, let z_n be the zeros of $\pi(z)$, let $S(r)=\bar{c}_\rho+\rho^{-1}\sum_{|z_n|\leqq r}\bar{z}_n^{-\rho}$ and let

$$C=\int_0^{2\pi}e^{i\rho\theta}dN(\theta),\quad q(\varphi)=-i\int^{2\pi}\theta e^{-i\rho\theta}dN(\varphi+\theta).$$

Then as $r\to\infty$,

$$r^{-\rho}\log f(re^{i\varphi})=q(\varphi)+\{\bar{C}(i\pi-\rho^{-1})+\overline{S(r)}\}e^{i\rho\varphi}+\epsilon(r,\varphi),$$

where $\limsup\Re\epsilon(r,\varphi)=0$, $\lim\Re\epsilon(r,\varphi)=0$ for each φ on an r-set of linear density 1 and

$$\limsup\Im\epsilon(r,\varphi)\leqq\pi\{N(\varphi+)-N(\varphi-)\}.$$

Interesting geometrical interpretations of this result are given in various special cases. As consequences of this principal theorem, improvements are given of other results of the author [same Comment. **14**, 314–349 (1942); **16**, 1–18 (1944); these Rev. **5**, 258]. The main result is also extended to functions whose zeros are measurable with respect to a proximate order. *R. P. Boas, Jr.* (Providence, R. I.).

***Mäder, Oskar. Über das asymptotische Verhalten meromorpher Funktionen bei speziell gegebener Null- und Polstellenverteilung.** Thesis, University of Freiburg, Switzerland, 1942. 27 pp.

The author extends to meromorphic functions some of the results of Pfluger [Comment. Math. Helv. **11**, 180–213 (1938)] for entire functions. Let $f(z)$ be a quotient of canonical products of nonintegral order and let all its zeros and poles be on the negative real axis. Let $n(r)$ denote the excess of zeros over poles in $|z|\leqq r$. In most of the thesis the author assumes that $n(r)\sim Dr^{\rho(r)}$, where $\rho(r)$ is a Lindelöf proximate order, $\rho(\infty)=\rho$, not an integer. Then

$$(*)\qquad r^{-\rho(r)}\log|f(re^{i\varphi})|\to\pi D\operatorname{cosec}\rho\pi\cos\rho\varphi$$

as $r\to\infty$, $|\varphi|<\pi$; this reduces to Pfluger's result if there are no poles. If the growth of $n(r)$ is further restricted, the growth of $|f(z)|$ can be specified also on the negative real axis away from the zeros; for example, (*) holds also for $\varphi=\pi$ except on a set of intervals $|r-r_n|\leqq|r_n|^{-\sigma}$, σ an arbitrary positive number, about the zeros and poles r_n, provided that $n(r)r^{-\rho}-D=o(1/\log r)$. Similar extensions are given for results of Pfluger for the case where the zeros and poles are regularly distributed in an angle.

R. P. Boas, Jr. (Providence, R. I.).

Delange, Hubert. Sur certaines fonctions entières. C. R. Acad. Sci. Paris **222**, 853–854 (1946). [MF 16283]

An entire function $f(z)$ is said to be oriented in the direction of a ray l from the origin if every sector with vertex at the origin and containing l has only a finite number of zeros exterior to it. Let $f(z)$ and $g(z)$ be entire functions of genus p, real on the real axis, and oriented in the direction of the negative real axis. For a value of θ, $|\theta|<\pi$, for which there exists no odd integer m satisfying $m\pi/(2p+2)\leqq|\theta|\leqq m\pi/2p$, as $r\to\infty$ one has

$$(1)\qquad \log|f(re^{i\theta})|\sim\log|g(re^{i\theta})|.$$

For any ϕ satisfying $\pi/(2p+2)<\phi<\pi/2p$, $f(z)$ and $g(z)$ exist of the type indicated such that (1) holds for $|\theta|=m\phi$, m an odd integer less than $2p$, and such that, for all other values of θ, $|\theta|<\pi$, $\log|f(re^{i\theta})|=o\{\log|g(re^{i\theta})|\}$. Let $f(0)=1$, $f^{(i)}(0)=0$, $i=1,2,\cdots,q-1$, and let all the zeros of $f(z)$ be real and negative, denoted by $-r_n$, $n=1,2,\cdots$, and arranged according to magnitude. If $r_n=r_{n+1}$, let $r_n'=r_n$, and, if $r_n<r_{n+1}$, let $-r_n'$ be the greatest zero of $f'(z)$ between $-r_{n+1}$ and $-r_n$. There will also be $s\leqq p$ supplementary zeros of $f'(z)$. Let $\bar{\omega}$ be the product of the moduli of those supplementary zeros which are not null. Let $\rho(x)$ satisfy

$$p<\liminf_{x\to+\infty}\rho(x)\leqq\limsup_{x\to+\infty}\rho(x)<p+1,$$

$\rho'(x)\cdot x\log x\to0$ as $x\to+\infty$. Then, among several results, the following is typical: $s=p$ and, as $x\to\infty$,

$$\sum_{r_i\leqq x}(r_i'-r_i)\sim\{p+1-\rho(x)\}x,$$

$$\bar{\omega}\prod_{r_i\leqq x}r_i'/r_i\sim q|a_q|x^{p+1-\rho(x)}/\rho(x).$$

M. S. Robertson (New Brunswick, N. J.).

Delange, Hubert. Sur certaines fonctions méromorphes. C. R. Acad. Sci. Paris **222**, 40–42 (1946). [MF 15978]

Let $f(z)=e^{G(z)}H(z;\alpha)/H(z;\beta)$, where $G(z)=A_0+A_1z+\cdots+A_pz^p$ and

$$H(z;\alpha)=\prod\left(1+\frac{z}{\alpha_j}\right)\exp\{-z/\alpha_j+\cdots+(-1)^pz^p/p\alpha_j^p\}.$$

The A's are real and the α's, β's are real and positive. Let $\nu(t)$ be the number of zeros of $f(z)$ of modulus not greater than t diminished by the number of poles of modulus not greater

than t. Let $\nu(t)$ be constantly of one sign. Let $\log f(z)$ represent the determination of $\log f(z)$ which is real for real and positive z. Let $|\theta|<\pi$. Let $\rho(x)$ be defined for $x>0$, bounded, differentiable and such that $\lim_{x\to+\infty}\rho'(x)\cdot x\log x=0$. The author states several results which include among others the following.

If, for a value of θ, as $r\to\infty$, $\log f(re^{i\theta})\sim Ar^{\rho(r)}e^{i\theta\rho(r)}$, this same relation holds for all the values of θ. This relation also holds if $\log|f(re^{i\theta})|\sim Ar^{\rho(r)}\rho(r)\cos\theta$ for a θ such that $|\theta|<\pi/(2p+2)$. The relation holds only if, for h an integer such that $-1\leqq h\leqq p$,

$$h\leqq\liminf\rho(x)\leqq\limsup\rho(x)\leqq h+1,\qquad x\to+\infty.$$

In order for the relation to hold for all θ it is necessary and sufficient that, as $x\to+\infty$,

$$\int_0^x\nu(t)dt\sim\frac{A}{\pi}x^{\rho(x)+1}\frac{\sin\pi\rho(x)}{\rho(x)+1},$$

$$A_q=(-1)^{q+1}\int_0^\infty t^{-q-1}\nu(t)dt,\qquad q\geqq h+1,$$

when

$$h<\liminf_{x\to+\infty}\rho(x)\leqq\limsup_{x\to+\infty}\rho(x)<h+1.$$

M. S. Robertson (New Brunswick, N. J.).

***Milloux, Henri. Les fonctions méromorphes et leurs dérivées. Extensions d'un théorème de M. R. Nevanlinna. Applications.** Actualités Sci. Ind., no. 888. Hermann et Cie., Paris, 1940. 53 pp.

This monograph is concerned with a comparative study of the values attained by a function $f(z)$ meromorphic in a given region D and those attained by a linear combination $\psi(z)$ of $f(z)$ and its first l derivatives, the coefficients being analytic and nonvanishing in D. The second fundamental theorem of the theory of meromorphic functions is extended in such a manner that f in the density index $N(r,1/f-1)$ is replaced by ψ. This extension calls for the introduction of a numerical coefficient in the term $N(r,f)$ which appears in the classical form of the second fundamental theorem.

Applications of this study include extensions of Landau's theorem, where now ψ rather than f itself is forbidden to attain the value one in the region concerned. In addition, a study is made of functions $f(z)$ which are meromorphic in the interior of the unit circle, which vanish at most n times and have at most p poles and for which the associated ψ attains the value one at most q times. It is shown for such f that, if $|f|<M$ in $|z|\leqq\frac{1}{2}$ on a set E which is not contained in a set of circles the sum of whose pseudo-radii is less than $2e/100$ (pseudo-distance of z_1 and z_2 in $|z|<1$ being defined as the absolute value of $(z_1-z_2)/(1-\bar{z}_1z_2)$, then, apart from a certain finite set of circles the sum of whose pseudo-radii is less than $2e/100$, $(1-|z|)\log|f(z)|$ is majorized by an explicitly given expression which depends upon M, n, p, q, l, $|z|$ and the coefficients of ψ. Related results are also given. *M. H. Heins* (Providence, R. I.).

Tumura, Yosiro. Sur le premier théorème dans la théorie des fonctions méromorphes. Proc. Imp. Acad. Tokyo 18, 164–169 (1942). [MF 14748]

A proof is given for the first fundamental theorem of the theory of meromorphic functions. Aspects of the problem of Kunugui [same Proc. 15, 27–32 (1939)] are discussed.

M. H. Heins (Providence, R. I.).

Kametani, Syunzi. The exceptional values of functions with the set of linear measure zero of essential singularities. II. Proc. Imp. Acad. Tokyo 19, 438–443 (1943). [MF 14840]

[For part I, see the same Proc. 17, 117–120 (1941); these Rev. 3, 78.] The following theorem is established. Let L denote a rectifiable Jordan arc in the finite w-plane, let w_0 denote a point not on L and let $V(L)$ denote the total variation of $\arg(w-w_0)$, $w\epsilon L$. It is assumed that $V(L)<+\infty$. If D is a region and E a compact subset of linear measure zero, and if $w=f(z)$ is regular and single-valued in $D-E$ and has E as its set of essential singularities, then the set S of all finite values not assumed by $f(z)$ in $D-E$ is such that its intersection with L is of linear measure zero. The proof is based upon the use of an auxiliary bounded analytic function associated with subsets of L of positive linear measure and a theorem of Besicovitch [Proc. London Math. Soc. (2) 32, 1–9 (1931)]. *M. H. Heins.*

Jørgensen, Vilhelm. Elementary proof of a theorem of the Picard type. Mat. Tidsskr. B. 1946, 129–134 (1946). (Danish) [MF 16314]

The Julia-Carathéodory lemma states that an analytic function $w=f(z)$, $z=x+iy$, $w=u+iv$, which is regular and satisfies $u>0$ in $x>0$ has always a nonnegative angular derivative $\lim u/x=c$, where z tends to ∞ in an angle less than π. The author proves that this lemma holds, except for the sign of c, under the sole condition that every real interval of fixed length $k>0$ contains a value which is not taken by w. The proof of this strong generalization is based on an application of a method of Ahlfors [Trans. Amer. Math. Soc. 43, 359–364 (1938)]. *L. Ahlfors.*

Nevanlinna, Rolf. Quadratisch integrierbare Differentiale auf einer Riemannschen Mannigfaltigkeit. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. 1, 34 pp. (1941). [MF 16497]

This paper represents a contribution to the problem of extending the classical theory of Abelian integrals to general abstract Riemann surfaces. The developments of the paper proceed on the assumptions that (1) the ideal boundary of the Riemann surface F is of capacity zero (null boundary) and (2) that the analytic differentials $dw(z)$ considered are single-valued and everywhere regular on F and in addition are such that the invariant Dirichlet integral

$$\int\!\!\int_F|dw/dz|^2dxdy$$

is finite ($dw(z)$ is then termed quadratically integrable). Thus the study is an extension of the classical theory of Abelian differentials of the first kind. The principal results of the paper are the following.

If F is a surface of finite genus p with null boundary, then there exist precisely p linearly independent differentials dw subject to the above restrictions which are of the first kind. If, in particular, F is planar, then $dw\equiv0$, independently of the connectivity of the surface.

If F is a surface with null boundary and of infinite genus, then there exists a denumerable infinity of linearly independent differentials of the first kind which are quadratically integrable. Furthermore, there exists a normalized orthogonal set of differentials $dw_\nu=f_\nu dz$ such that the totality of admitted differentials of the first kind $dw=fdz$ are represented by the convergent series $dw=\sum c_\nu dw_\nu$ with $\sum_1^\infty|c_\nu|^2<+\infty$. The square of the norm of the co-

variant analytic vector f is defined by $\iint_F |f|^2 dxdy$ and $c_\nu = \iint_F f\bar{f}_\nu dxdy$.

Use is made of the normalized representation of an abstract Riemann surface due to the author [Ann. Acad. Sci. Fennicae. Ser. A. **54**, no. 3 (1940); these Rev. **2**, 85]. A series of lemmas is developed leading to the following lemma which plays a fundamental role in the argument. If the differential $dw = du + idv$ on a Riemann surface F with null boundary is regular and quadratically integrable on F, and if u is single-valued on F, then $dw \equiv 0$. From this result it is deduced that, if $dw = du + idv$ is a single-valued regular quadratically integrable differential on F (F having a null boundary) and if the periodicity moduli of u vanish for all retrosections of F which do not divide F and which belong to a fundamental system, then $dw \equiv 0$ on F. This result is decisive in establishing the completeness of the normal orthogonal set of differentials that is introduced. A system of elementary differentials is set up along classical lines, each being associated with a nondividing retrosection of F. The system so obtained is orthogonalized and normalized, and it is then shown that the set so obtained is complete by typical Hilbert space arguments combined with the uniqueness theorem cited above.

The paper is concluded with remarks concerning further developments of the theory and applications to the theory of automorphic functions. *M. H. Heins.*

Myrberg, P. J. Über transzendente hyperelliptische Integrale erster Gattung. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. **14**, 32 pp. (1943). [MF 15211]

This paper is concerned with the construction of a theory paralleling that of hyperelliptic integrals for two-sheeted Riemann surfaces with infinitely many branch points all of which are real and, in particular, with the determination of the integrals by their periods. The present investigation represents the first part of the author's study and is directed toward integrals of the first category. In this study a fundamental role is played by certain special integral functions (Integralfunktionen) whose Dirichlet integrals over the whole Riemann surface are finite. Several examples are studied in detail. *M. H. Heins* (Providence, R. I.).

Myrberg, P. J. Über Integrale auf transzendenten symmetrischen Riemannschen Flächen. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. **31**, 21 pp. (1945). [MF 15212]

An extension is given of the theory of Abelian integrals for Riemann surfaces which are either orthosymmetric or diasymmetric, and have a null boundary. These surfaces are uniformized with the aid of functions automorphic with respect to a group of real linear fractional transformations which remains properly discontinuous on certain intervals of the real axis. With the aid of Θ-functions of dimension -2 which satisfy a system of simple functional equations, a system of integrals is introduced. These integrals serve as the analogues of the normalized elementary integrals of the three categories in the classical Abelian integral theory. Uniqueness properties are then deduced from the hypothesis of the null boundary. *M. H. Heins* (Providence, R. I.).

***Tornehave, Hans. Om regulære periodiske Funktioner af flere Variable.** [**On Regular Periodic Functions of Several Variables**]. Thesis, University of Copenhagen, 1944. 107 pp. (Danish)

Let $g(u_1, \cdots, u_m)$ be a regular function of m variables $u_\nu = \xi_\nu + iv_\nu$ in a domain determined by $(\xi_1, \cdots, \xi_m)\epsilon\Xi$, where Ξ is a domain in the real m-dimensional space, and suppose that $g(u_1, \cdots, u_m)$ has the period $2\pi i$ in each variable. The Jensen formula for functions of one variable suggests a connection between the distribution of the zeros of $g(u_1, \cdots, u_m)$ and the "Jensen function"

$$\psi(\xi_1, \cdots, \xi_m) = (2\pi)^{-m} \int_0^{2\pi} \cdots \int_0^{2\pi} \log |g(\xi_1 + iv_1, \cdots, \xi_m + iv_m)| dv_1 \cdots dv_m.$$

The present thesis contains a detailed study of this question.

In chapters I–III an account is given of the theory of analytic functions of m variables. New is the following theorem, communicated by C. L. Siegel: if $f(z_1, \cdots, z_m, w)$ is regular for $(z_1, \cdots, z_m)\epsilon Z$ and $w\epsilon W$, where Z and W are closed bounded sets, and if $N(z_1, \cdots, z_m)$ denotes the number of zeros of $f(z_1, \cdots, z_m, w)$ in W, when the function is not identically zero, and otherwise 0, then $N(z_1, \cdots, z_m)$ is bounded in Z.

In chapter IV the Jensen function is introduced. It is proved that it is a continuous convex function in Ξ and that it is linear in a domain Ξ_0 if and only if $g(u_1, \cdots, u_m)$ has no zeros in the domain determined by $(\xi_1, \cdots, \xi_m)\epsilon\Xi_0$. It is nondifferentiable at a point $(\xi_1^{(0)}, \cdots, \xi_m^{(0)})$ if and only if the manifold $\xi_1 = \xi_1^{(0)}, \cdots, \xi_m = \xi_m^{(0)}$ contains an $(m-1)$-dimensional manifold of zeros of $g(u_1, \cdots, u_m)$. If $g(u_1, \cdots, u_m)$ is an entire function and is 2π-irreducible (that is, is not the product of two entire functions with period $2\pi i$ in each variable both of which have zeros), the nondifferentiability of $\psi(\xi_1, \cdots, \xi_m)$ in $(\xi_1^{(0)}, \cdots, \xi_m^{(0)})$ implies a symmetry-property, namely,

$$\overline{g(2\xi_1^{(0)} - \bar{u}_1, \cdots, 2\xi_m^{(0)} - \bar{u}_m)} = g(u_1, \cdots, u_m) h(u_1, \cdots, u_m),$$

where $h(u_1, \cdots, u_m)$ is an entire function without zeros. This implies that the points of nondifferentiability form linear manifolds of a certain dimension p $(0 \leqq p < m)$ arranged in a simple manner. For a 2π-irreducible exponential polynomial the nondifferentiability points (if any) form one such manifold.

Results are also obtained on the distribution of the points of nondifferentiability when $g(u_1, \cdots, u_m)$ is not an entire function. The mean derivatives of $\psi(\xi_1, \cdots, \xi_m)$ at such a point in different directions define a linear function, that is, there exist numbers $\alpha_1, \cdots, \alpha_m$ such that, for arbitrary $\lambda_1, \cdots, \lambda_m$,

$$\lim_{h \to 0} (2h)^{-1} \{\psi(\xi_1^{(0)} + h\lambda_1, \cdots, \xi_m^{(0)} + h\lambda_m) - \psi(\xi_1^{(0)} - h\lambda_1, \cdots, \xi_m^{(0)} - h\lambda_m)\} = \alpha_1\lambda_1 + \cdots + \alpha_m\lambda_m.$$

In chapter V certain integral relations are proved connecting the measure of the zero-point manifold of $g(u_1, \cdots, u_m)$ with the derivatives of $\psi(\xi_1, \cdots, \xi_m)$. These relations are generalizations of Jensen's formula. One of them had previously been announced [C. R. Acad. Sci. Paris **208**, 784–786 (1939)].

Chapter VI contains examples and chapter VII contains applications to analytic almost periodic functions of the form $f(s) = g(\lambda_1 s, \cdots, \lambda_m s)$ which complete results on mean motions and zeros of analytic almost periodic functions obtained in a joint paper by the reviewer and the author [Acta Math. **77**, 137–279 (1945); these Rev. **7**, 438]. *B. Jessen* (Copenhagen).

Hua, Loo-keng. On the theory of Fuchsian functions of several variables. Ann. of Math. (2) 47, 167–191 (1946). [MF 16328]

Suppose that R is a bounded region of the $2n$-dimensional space given by $z=(\cdots, z_k, \cdots)$, $z_k=x_k+iy_k$, $1\leqq k\leqq n$. Let $0\epsilon R$. Assume that the group Γ of analytic automorphisms of R is transitive. The author develops a theory of Poincaré theta series and Fuchsian forms for certain discontinuous subgroups G of Γ. To furnish the usual existence proofs for a fundamental region $F\subset R$ for G an invariant metric $\Delta(a, b)$ with $a, b\epsilon R$ (with respect to Γ) is introduced as follows. Since Γ_0, the subgroup of Γ with the fixed point 0, is compact, each $\gamma_0\epsilon\Gamma_0$ can be described by a unitary matrix $U=(u_{ij})$ such that $w=\gamma_0(z)=t_{0,U}(z)=(\cdots, \sum_{j=1}^n u_{ij}z_j+\text{higher terms}, \cdots)$. Then each $\gamma\epsilon\Gamma$ can be expressed as $w=\gamma(z)=t_{a,U}(z)=t_{a,I}[t_{0,U}(z)]$, where $t_{a,I}$ is the transformation carrying 0 to a suitable interior point $a\epsilon R$. Let $J_{a,U}(z)=(\partial w_i/\partial z_j)$. Then

$$\overline{(dz)J_{a,U}(z)}J_{a,U}(z)'(dz)'=\overline{(dz)}H(z,\bar{z})(dz)'$$

is proved to be the essentially unique nonsingular Hermitian differential form for Γ in case Γ_0 is irreducible. Next $\Delta(a, b)=\text{g.l.b. } |\int_C\{(d\bar{z})H(z,\bar{z})(dz)'\}^{\frac{1}{2}}|$ for all rectifiable curves C of R connecting a, b. With the proper modifications, the classical methods for constructing series and forms with given multipliers for G can now be applied. Since G is necessarily properly discontinuous and denumerable, the Poincaré series $\Theta_k(z)=\sum(\partial S(z)/\partial z)^k$ for $S\epsilon G$ can be formed. The series converges absolutely (and uniformly) for $k\geqq 2\lambda$ if $\int_R d(H(z,\bar{z}))^{1-\lambda}dx_1\cdots dy_n$ converges. As usual, convergence is assured for $k\geqq 2$. Next, Siegel's method can be employed, with a few minor modifications, to show that the Fuchsian functions for R for G form an algebraic function field of n variables provided R is a circular region and F is compact. The author indicates how the latter result may be established by means of Blumenthal's method if R is not circular. It is shown how the validity of a generalized "Verzerrungssatz" implies more specific theorems. Finally, the geometry of complex spheres R determined by $|zz'|^2+1-2\bar{z}z'>0$ and $|zz'|<1$ is presented in some detail. The "Verzerrungssatz" holds in this case. The space R is circular and has a non-positive Riemann curvature. Then $\Delta(a, b)$ equals the length of the unique geodesic between a and b; using this result, considerable improvements of the general proof can be made.

O. F. G. Schilling (Chicago, Ill.).

Tôyama, Hiraku. Zur Theorie der hyperabelschen Funktionen. Proc. Imp. Acad. Tokyo 19, 415–419 (1943). [MF 14834]

Tôyama, Hiraku. Zur Theorie der hyperabelschen Funktionen. II. Proc. Imp. Acad. Tokyo 20, 554–557 (1944). [MF 14923]

Tôyama, Hiraku. Zur Theorie der hyperabelschen Funktionen. III. Proc. Imp. Acad. Tokyo 20, 558–560 (1944). [MF 14924]

The fundamental group F of a Riemann surface of genus $p>1$ which is punctured at l points at which finite signatures are prescribed is determined by $2p+l$ generators a_i, b_i, c_μ subject to the relations

$$\prod_{i=1}^p a_ib_ia_i^{-1}b_i^{-1}\prod_{\mu=1}^l c_\mu=1,$$

$c_\mu^{e_\mu}=1$. These papers provide proofs for some results on the unitary representations of F stated by A. Weil [J. Math. Pures Appl. (9) 17, 47–87 (1938)]. The first problem is concerned with the dimension d of the manifold M of all representations of degree $r>1$ of F. It is shown that d equals

$$r^2(2p-1)+1+2\sum_\mu\sum_{\alpha<\beta}N_{\mu\alpha}N_{\mu\beta},$$

where $N_{\mu\alpha}$ is the number of characteristic values of the representing matrix C_μ of c_μ. Furthermore, the complex dimension d coincides with the topological dimension of M. The proofs involve an algebraic reduction of certain matrices to diagonal form and the application of the intersection theory of topology. The methods can also be applied to evaluate the dimensions of classes of representations. Moreover, it is shown that each class of representations contains unitary representations. Finally, the analogue of Jacobi's inversion theorem for Abelian functions is formulated as a duality theorem in the sense of Tannaka [see C. Chevalley, Theory of Lie Groups, I, Princeton University Press, 1946, p. 211; these Rev. 7, 412]. Thus F is identified with a suitably restricted group of continuous mappings on matrix systems of the unitary representations.

O. F. G. Schilling (Chicago, Ill.).

Krasner, Marc. Essai d'une théorie des fonctions analytiques dans les corps valués complets: séries de Taylor et de Laurent issues de ces corps. C. R. Acad. Sci. Paris 222, 37–40 (1946). [MF 15977]

Krasner, Marc. Essai d'une théorie des fonctions analytiques dans les corps valués complets: fonctions holomorphes et méromorphes. C. R. Acad. Sci. Paris 222, 165–167 (1946). [MF 15988]

Krasner, Marc. Essai d'une théorie des fonctions analytiques dans les corps valués complets: théorèmes de Nevanlinna; transformations holomorphes. C. R. Acad. Sci. Paris 222, 363–365 (1946). [MF 16013]

Krasner, Marc. Essai d'une théorie des fonctions analytiques dans les corps valués complets; transformations holomorphes et leurs applications algébriques; fonctions holomorphes de plusieurs variables et fonctions implicites; familles normales; prolongement analytique. C. R. Acad. Sci. Paris 222, 581–583 (1946). [MF 16036]

In these notes the author develops the theory of analytic functions in a complete valuation field and makes algebraic applications of this theory and of his previous results.

Let n run through all integral values and let M be any set of points Q_n in the real plane $O\xi\eta$ with coordinates $\xi(Q_n)=n$ and $\eta(Q_n)$. The lower convex envelope of M is the Newton polygon Π of M. Let P_0 be the intersection of Π with $O\eta$ if this exists; let $P_1, P_2, \cdots, P_{s^*}$ be the vertices of Π for ξ positive and $P_{-1}, P_{-2}, \cdots, P_{-s'+1}$ be those for ξ negative; let $L_i=P_iP_{i+1}$, and $-\nu_i$ the slope of L_i. Let $\bar{\nu}'$ be the upper bound of ν_{-i} $(0<i\leqq s')$ and $\bar{\nu}^*$ be the lower bound of ν_i $(0\leqq i\leqq s^*)$, these bounds being taken on the semi-real number system [Krasner, same C. R. 219, 433–435 (1944); these Rev. 7, 364]. A ν-tangent to M is a straight line of slope $-\nu$, where $\bar{\nu}^*\leqq\nu\leqq\bar{\nu}'$, which is tangent to Π in the sense that it either coincides with a side or passes through a vertex of Π in such a way that the two sides meeting in the vertex are on the same side of the ν-tangent. A ν-tangent is unique. If $\varphi(\Pi;\nu)$ is the η-intercept of the ν-tangent to Π then $\varphi(\Pi;\nu)$ is a continuous function of ν, linear by segments, and convex increasing, constant or decreasing according as $\nu\leqq\nu_0$, $\nu_0\leqq\nu\leqq\nu_{-1}$, or $\nu_{-1}\leqq\nu$.

Let k be a field complete with respect to a valuation $|\cdots|$ and K an extension of k, the valuation $|\cdots|$ also being extended to K. Assume that K is not locally compact

and that it contains the completion of every simple subextension of K/k. An element ∞_K with neighborhoods $|x|>r$, r any real number, is adjoined to K. A function defined in K and having values in K is called a Laurent k-series in $x-\alpha$ if it has the form $\sum_{-\infty}^{+\infty} a_n(x-\alpha)^n$, $\alpha \epsilon K$ and $a_n \epsilon k$. It is a Taylor series if $a_n=0$ for $n<0$. A function $f(x)$ is holomorphic in an open set $\mathfrak{D}$ of K if there exists a Laurent series which converges to $f(x)$ everywhere in $\mathfrak{D}$.

Let $f(x)=\sum a_n(x-\alpha)^n$ be a Laurent k-series and let $|\cdots|$ and $\omega(\cdots)=-\log|\cdots|$ denote the valuation and order in K. Let $M(f)$ be the set of points Q_n with coordinates $\xi(Q_n)=n$, $\eta(Q_n)=\omega(a_n)$. The Newton polygon Π of $M(f)$ is called the Newton polygon of $f(x)$. The domain of convergence of $f(x)$ is then $r'<|x-\alpha|<r^*$, where $\log r'=-\nu'$ and $\log r^*=-\nu^*$ except in certain stated circumstances when $f(x)$ converges also for $|x-\alpha|=r'$ and/or $|x-\alpha|=r^*$. Also $-\log|f(x)|=\omega[f(x)]\geqq \min \omega[a_n(x-\alpha)^n]=\varphi(\Pi;\nu)$, where $\nu=\omega(x-\alpha)$. The sign of equality holds if the minimum is attained for one term only, which occurs if ν is not among the ν_i. If $\nu=\nu_i$ all terms T_n for which $\omega(T_n)$ obtains its minimum correspond to points on the side L_i of Π. Letting $\bar{\beta}$ represent the element of the skeleton S of K [see Krasner, same C. R. **219**, 345–347 (1944); these Rev. **7**, 363] to which β belongs, and writing $\xi=\overline{x-\alpha}$ and $\xi^{\lambda_i}f_i(\xi)=\sum \bar{a}_n(\overline{x-\alpha})^n$, where the summation extends over all terms T_n for which $\omega(T_n)$ attains its minimum, the function $\Phi_\alpha(\xi)$ is defined to be $\xi^{\lambda_i}f_i(\xi)$ or $\bar{a}_{\lambda_i}\xi^{\lambda_i}$ according as $|\xi|=r_i$ or $r_{i-1}<|\xi|<r_i$. This function is of importance in later applications.

Let $M(r)$ and $M_\epsilon(r)$ be the upper bounds of $|f(x)|$ for $x\epsilon K$, on the circle $|x-\alpha|=r$ and in the ring $r-\epsilon<|x-\alpha|<r+\epsilon$, respectively. The possible points of discontinuity of $M(r)$ are $r_i=\exp(-\nu_i)$, which form a discrete set on the open interval (r', r^*). At all points of continuity $\log M(r)=-\varphi(\Pi; -\log r)$. At all points of (r', r^*), however, $M^*(r)=\lim_{\epsilon\to 0+} M(r)$ exists and is equal to $-\varphi(\Pi; -\log r)$. Replacing $M(r)$ by $M^*(r)$, the Hadamard three circle theorem follows as does the Cauchy inequality $|a_n|r^n\leqq M^*(r)$ and the Weierstrass theorem that the limit of a sequence of functions holomorphic in a circle C is a holomorphic function in C. There follows a discussion of meromorphic functions, position of zeros, and integral functions. In the case that K is algebraically closed analogues of several well-known results are obtained, including the theorems of Weierstrass and Picard on integral functions and the first and second theorems of Nevanlinna.

Let $\psi(\xi)$ be a polynomial in S and Z_ψ the set of all elements ξ of S for which $\psi(\xi)$ has a meaning (that is, all its terms are addible). If $Z\subseteq Z_\psi$ is a ring $\bar{\rho}\leqq|\xi|\leqq\rho$ in S, the set of values of $\psi(\xi)$, $\xi\epsilon Z$, is denoted by $(Z;\psi)$ and is called a polynomial retract of S. The union Λ of polynomial retracts is called a retractive set of S. If $\psi(\xi)$ and S are such that there exists a subcorpoid S' of S containing the coefficients of $\psi(\xi)$ and such that the field R of S is a pure transcendental extension of the field R' of S', then the retract $(Z;\psi)$ is said to be decomposable by means of S'. A retractive set Λ is decomposable if it is the union of retracts all decomposable by means of the same corpoid S'. A decomposable retract $(Z;\psi(\xi))$ defines $\psi(\xi)$ up to a linear transformation $\xi'=\alpha\xi+\beta$ $(\alpha, \beta\epsilon S')$ and a decomposable retractive set completely defines its minimal retracts.

There follows a discussion of holomorphic transformations of the form $T: x\to F(x)$, where $F(x)$ is a Taylor series with circle of convergence C^*. If $A\subseteq K$ the set $S(A)$ consists of all elements $\bar{\alpha}$ of S, where $\alpha\epsilon A$. Then $S(TC^*-F(\alpha))$, denoted by $S_\alpha(T;K)$, is a retractive set of S and its decomposition is discussed. It is decomposable if there exists a field K' between k and K such that R is a pure transcendental extension of its field of residues R'.

Algebraic applications are then considered. If α is a simple zero of an irreducible polynomial $F(x)$ in a field k^*, and $k=k^*(\alpha)$, the Newton polygon Π and skeleton function $\Phi_\alpha(\xi)$ are constructed for $F(x)$. These determine the structural properties of $S_\alpha(T;K)$, where K is the union of k and a field in which $F(x)$ remains irreducible. These are in turn linked with the ramification properties of k/k^*. The manner in which $S_\alpha(T;K)$ decomposes also indicates whether or not k/k^* is a Galois extension.

Finally, there are some remarks on Taylor and Laurent expansions of functions of several variables, corresponding Newton polytopes, implicit functions and analytic continuation. *D. C. Murdoch* (Vancouver, B. C.).

Häfeli, Hans. Quaternionengeometrie und das Abbildungsproblem der regulären Quaternionenfunktionen. Comment. Math. Helv. 17, 135–164 (1945).

Der erste, aus den §§ 1–3 bestehende Teil der Arbeit ist vorbereitender Natur, indem in den beiden ersten §§ das später gebrauchte quaternionengeometrische Werkzeug bereitgestellt wird, während § 3 eine kurze Rekapitulation der Definition der regulären Quaternionenfunktionen enthält. Der zweite Teil, §§ 4–6, handelt von den geometrischen Eigenschaften der regulären Quaternionenfunktionen, wobei die leitende Frage die nach dem Zusammenhang der regulären Abbildungen des Quaternionenraumes mit den konformen ist.

In § 4 ist die Frage nach denjenigen regulären Abbildungen eines infinitesimalen Gebietes des Quaternionenraumes gestellt, die zugleich konform sind. Die in den Sätzen 1 und 2 enthaltene Antwort sagt aus, dass jede rechts- (links-) reguläre Abbildung eines infinitesimalen Gebietes, die zugleich konform ist, eine uneigentliche Drehstreckung ist, wobei die Drehung rechts- (links-) seitig von 90° ist. In § 5 ist gezeigt (Sätze 3 und 4), dass jede in einem infinitesimalen Gebiet rechts- (links-) reguläre Funktion sich in diesem Gebiet als Summe dreier rechts- (links-) regulärer Funktionen darstellen lässt, von denen jede eine konforme Abbildung des Gebietes vermittelt. Mit den Sätzen 1–4 hat man den vollständigen Überblick über die regulären Abbildungen eines infinitesimalen Gebietes.

Der abschliessende § 6 behandelt die Frage nach denjenigen in einem endlichen Gebiet regulären Funktionen, die das ganze Gebiet konform abbilden. Die Antwort ist in Satz 5 ausgesprochen und heisst, dass eine solche Funktion immer von der Form $w=m\bar{z}n+l$ ist, wobei der Realteil von m (n) verschwindet, wenn die Funktion rechts- (links-) regulär ist. *W. Nef* (Fribourg).

Bourion, G. Fonctions quasi analytiques (P) dans le champ complexe. Bull. Sci. Math. (2) 69, 137–148 (1945). [MF 15890]

Soit C un continu ne morcelant pas le plan. Une fonction $f(x)$ définie sur C est dite quasi-analytique (P) sur C relativement à la suite d'exposants $\{n_k\}$, s'il existe un nombre $0<\delta<1$, tel que, quel que soit $n\epsilon\{n_k\}$, il existe un polynome $P_n(x)$ de degré n satisfaisant, sur C, l'inégalité $|f(x)-P_n(x)|<\delta^n$. Cette notion de quasi-analyticité généralise celle de S. Bernstein pour un segment. L'auteur démontre plusieurs théorèmes concernant la quasi-analyticité sur un continu. Ainsi, par exemple, les sommes partielles de

degré n_k du "développement en série de polynomes de Faber associé à $f(x)$" convergent vers cette function et la convergence est "quasi-analytique." *S. Mandelbrojt.*

Theory of Series

Cattaneo, Paolo. Sui numeri di Fibonacci. Boll. Un. Mat. Ital. (2) 5, 196–199 (1943). [MF 16103]

Sibirani, F. Fonti di identità numeriche. Boll. Un. Mat. Ital. (2) 4, 187–199 (1942). [MF 16070]

By transformations of integrals the author derives a considerable number of algebraic identities; some of these are new. *I. Kaplansky* (Princeton, N. J.).

Juzuk, D., and Motzkin, Th. A multiplicatory formula for the general recurring sequence of order 2. Math. Student 13, 61–63 (1945). [MF 15647]

The authors consider a generalization of a theorem of Siebeck [J. Reine Angew. Math. 33, 71–77 (1846)] concerning recurring series of the second order. Let a, b, w_0, w_1 be arbitrary complex numbers and define the sequence w_n by $w_n=aw_{n-2}+bw_{n-1}$; then w_{kl} may be expressed as a linear combination of $w_0, w_1, \cdots, w_k$,

$$w_{kl}=\sum_{i=0}^{k}\binom{k}{i}(au_{l-1})^{k-i}u_l^i w_i,$$

where u_n is the special case of w_n in which $w_0=u_0=0$, $w_1=u_1=1$. *D. H. Lehmer* (Berkeley, Calif.).

Wright, E. M. On a sequence defined by a non-linear recurrence formula. J. London Math. Soc. 20, 68–73 (1945). [MF 16697]

The sequence is defined for $\alpha>0$ by the recurrence formula $(n-1)c_n=\sum_{m=1}^{n-1}c_m c_{n-m}\exp\{-(m-1)\alpha\}$ for $n\geqq 2$, with $c_1=1$. To consider the behavior of c_n as $n\to\infty$, define the step function $j(t)$ as $j(t)=0$ $(1\leqq t<e)$, $j(t)=j(\log t)+1$ $(t\geqq e)$, which is such that $j(n)$ tends to infinity slower than any iterated logarithm of n. For any $\epsilon>0$, there are numbers k_1 and k_2, which depend only on ϵ and α, such that

$$c_n\leqq\exp\{k_1 n-(\tfrac{1}{2}\alpha-\epsilon)nj(n)\}, \tag{1}$$

and $c_n\geqq\exp\{k_2 n-(\alpha+1+\epsilon)nj(n)\}$.

The inequality (1) shows that the function $f(z)=\sum_{n=1}^{\infty}c_n z^n$ is an integral function of infinite order. The behavior of the function $f(z)$ is intimately related to the solution of the nonlinear difference-differential equation

$$\psi'(t+1)=-\alpha\psi(t)\{1+\psi(t+1)\}.$$

P. Civin (Eugene, Ore.).

Arbault, Jean. Sur les séries à termes positifs. C. R. Acad. Sci. Paris 222, 217–219 (1946). [MF 15994]

If $\sum\rho_n$ is a convergent series with positive terms, then $\sum_1^n 1/\rho_k\neq O(n^2)$. *A. Zygmund* (Philadelphia, Pa.).

Moore, E. H. Classes of sequences of positive numbers. Bull. Amer. Math. Soc. 52, 192–219 (1946). [MF 15624]

This paper was prepared for publication, by H. H. Goldstine and T. H. Hildebrandt, from notes and an incomplete manuscript written by E. H. Moore. It is a systematic determination of relations among numerous classes of sequences of positive numbers. It contains a multiplication table, giving 312 such relations, and numerous theorems involving relations among classes.

Let V denote the class of all sequences $a_1, a_2, a_3, \cdots$ of positive numbers. An element of V belongs to F if $|a_n|<M$, to L_0 if $a_n\to 0$, to L_∞ if $a_n\to\infty$, to C if $\sum a_n<\infty$ and to D if $\sum a_n=\infty$. If $P\subset V$ and p is a real positive or negative number, then P_p is the class of sequences $a_1, a_2, \cdots$ for which $a_1^p, a_2^p, \cdots$ is in P. If $P_f\subset P_e$ whenever $0<f<e$, then $P_e'=\sum_{0<f<e}P_f$ and $P_e''=\prod_{f>e}P_f$. If $P_f\supset P_e$ whenever $0<f<e$, then $P_e'=\sum_{f>e}P_f$ and $P_e''=\prod_{0<f<e}P_f$. These and analogous definitions give meaning to the symbols

$$\begin{array}{l} 0,\ C_0'',\ C_e',\ C_e,\ C_e'',\ C_f',\ C_f,\ C_f'', \\ C_\infty',\ L_0,\ F(-F_{-1}),\ F \text{ or } -F_{-1}, \\ F+(-F_{-1}),\ -L_\infty,\ D_{-\infty}'',\ D_{-f}', \\ D_{-f},\ D_{-f}'',\ D_{-e}',\ D_{-e},\ D_{-e}'',\ D_{-0}',\ V; \end{array} \tag{1}$$

when $0<e<f$, they are subclasses of V increasing in the order given from the null set to V. The cross product $P\times Q$ of two subclasses P and Q of V is the class of sequences representable in the form $a_1b_1, a_2b_2, \cdots$ with $a_1, a_2, \cdots$ in P and $b_1, b_2, \cdots$ in Q. The quotient P/Q is the class of sequences $c_1, c_2, \cdots$ in V such that $c_1b_1, c_2b_2, \cdots$ is in P whenever $b_1, b_2, \cdots$ is in Q. A major interest of the paper lies in such algebraic formulas as

$$P/Q=-(Q\times -P_{-1})_{-1}, \tag{2}$$

and such theorems as those saying that $P/Q\supset R$ is equivalent to $P\supset R\times Q$ while the reversed relation $P/Q\subset R$ does not always imply $P\subset R\times Q$. The multiplication table referred to above gives, for each pair P and Q of classes in (1), a class R in (1) such that $P\times Q=R$; when $P\neq Q$, not both $P\times Q$ and $Q\times P$ are tabulated since $P\times Q=Q\times P$. A division table can be formed by use of the multiplication table, the formula (2) and the classes

$$V,\ D_0',\ D_e'',\ D_e,\ \cdots,\ 0 \tag{3}$$

which are, in order, the complements of the classes in (1). If P and Q are classes in (1) and (3), respectively, then $P\times Q=V$ unless at least one of P and Q is null. The last third of the paper establishes conditions for validity of the formula $(P-Q)\times(R-S)=(P\times R)(-Q\times -S)$.

R. P. Agnew (Ithaca, N. Y.).

Kronrode, A. Sur les permutations des termes dans les séries numériques. C. R. (Doklady) Acad. Sci. URSS (N.S.) 49, 163–165 (1945). [MF 16404]

Let P be a permutation carrying the positive integers $1, 2, 3, \cdots$ into a rearrangement $n_1, n_2, n_3, \cdots$. The following and other classes of permutations are defined and related. If P is such that $\sum a(n_k)$ converges to s whenever $\sum a(n)$ is a series converging to s and $\sum a(n_k)$ diverges whenever $\sum a(n)$ diverges, then P is neutral. If there exists a convergent (divergent) series $\sum a(n)$ such that $\sum a(n_k)$ diverges (converges), then P is gauche (droite). If there exists a convergent series $\sum a(n)$ such that $\sum a(n_k)$ converges to a different value, then P is essential. The principal theorem which motivates the paper is the following. Each essential permutation is both gauche and droite; that is, if $\sum a(n)$ converges and $\sum a(n_k)$ converges to a different value, then there exists a convergent (divergent) series $\sum b(n)$ such that $\sum b(n_k)$ diverges (converges). Eleven theorems are announced. There are no proofs. [Cf. the following review.]

R. P. Agnew (Ithaca, N. Y.).

Kronrod, A. On permutation of terms of numerical series. Rec. Math. [Mat. Sbornik] N.S. 18(60), 237–280 (1946). (Russian. English summary)

Proofs of the results announced in the paper reviewed above. *R. P. Agnew* (Ithaca, N. Y.).

Hadwiger, H. Ein Satz über bedingt konvergente Vektorreihen. Math. Z. 47, 663–668 (1942). [MF 15909]

Let $a_1+a_2+\cdots$ be a convergent series of vectors, in an s-dimensional space E_s, for which the series $\sum|a_n|$ of moduli is divergent. Let E_t denote the t-dimensional ($1\leqq t\leqq s$) Steinitzian subspace of values of convergent rearrangements of $\sum a_n$. Let F be a closed convex subset of E_t. Then there is a rearrangement $\sum b_n$ of $\sum a_n$ such that the set of limit points of the sequence of partial sums of $\sum b_n$ is identical with the set F. *R. P. Agnew* (Ithaca, N. Y.).

Schieldrop, Edgar B. Des suites adjointes et des séries adjointes à une série donnée. Avh. Norske Vid. Akad. Oslo. I. 1944, no. 4, 19 pp. (1945). [MF 16439]

Let $a_1+a_2+\cdots$ be a series with partial sums $s_0=0$, $s_1=a_1$, $s_2=a_1+a_2$, $\cdots$. Then $\sum a_n$ converges if and only if there exist positive numbers σ, q_1, q_2, $\cdots$ such that

$$(1)\qquad q_na_n=\sigma-s_{n-1},\qquad n=1,2,\cdots.$$

If (1) holds, then

$$(2)\qquad a_{n+1}/a_n=(q_n-1)/q_{n+1};$$

if (2) holds, then (1) holds with $\sigma=a_1q_1$. A sequence q_n such that $q_n>0$ and (2) holds is called an adjoint sequence of $\sum a_n$, and the series $\sum q_n^{-1}$ is called an adjoint series of $\sum a_n$. [It is not pointed out by the author that if $\sum a_n$ converges to s then, corresponding to each $\sigma\geqq s$, there is exactly one adjoint sequence q_n defined by (1). By use of this observation, several of the author's theorems become obvious; for example, if $\sum a_n$ converges to s, then $\sum a_n$ has an infinite set of adjoints (one for each $\sigma>s$) and a unique one, the principal adjoint, for which $q_na_n\rightarrow 0$ (the one for which $\sigma=s$).]

By use of formulas derived from (2) and of infinite products, proof is given of the elementary facts that if $\sum a_n$ converges to s then $\sum a_n/(\sigma-s_{n-1})$ converges when $\sigma>s$ and diverges when $\sigma<s$. [The first follows from $a_n/(\sigma-s_{n-1})\leqq a_n/(\sigma-s)$ and the second from

$$\sum_{n=p}^{q}a_n/(s-s_{n-1})\geqq(s_q-s_{p-1})/(s-s_{p-1}).]$$

If $q_n>1$, $n=1,2,\cdots$, then to each positive a_1 corresponds a unique series $\sum a_n$ of positive terms such that (2) holds and hence (1) holds with $\sigma=q_1a_1$. Each such series $\sum a_n$ is convergent because s_n is increasing and $s_n=q_1a_1-q_{n+1}a_{n+1}<q_1a_1$. Numerical examples are given. *R. P. Agnew.*

Sales, Francisco. On some schemes of convergence. Revista Mat. Hisp.-Amer. (4) 5, 255–259 (1945). (Spanish) [MF 15886]

Let a_{nk}, $0\leqq k\leqq n$, and b_{nk}, $0\leqq k\leqq n$, be matrices of nonnegative numbers such that $a_{n0}+a_{n1}+\cdots+a_{nn}=1$, $n=0,1,2,\cdots$, and $b_{nk}<M$, $0\leqq k\leqq n<\infty$. Let

$$y_n=a_{n0}b_{n0}+a_{n1}b_{n1}+\cdots+a_{nn}b_{nn}.$$

It is asserted that, if $\lim_{n\to\infty}a_{nk}=a_k>0$, $\lim_{n\to\infty}b_{nk}=b_k$, then $\lim y_n=a_0b_0+a_1b_1+\cdots$, and that, if $\lim_{n\to\infty}a_{nk}=0$, $\lim_{n\to\infty}b_{nk}=b_k$, $\lim_{n\to\infty}b_k=b$, then $\lim y_n=b$. The assertions are erroneous. The supporting arguments use hypotheses (uniformity of limits) beyond those stated.

R. P. Agnew (Ithaca, N. Y.).

Knopp, Konrad. Eine Bemerkung zur C_k- und A-Limitierung von Funktionen. Math. Z. 50, 155–160 (1944). [MF 15845]

Let S denote the class of functions $s(t)$ defined, bounded and Lebesgue integrable over $0<t<c$ for each $c>0$. Let, when the integrals exist,

$$C_k(x;s(t))=kx^{-k}\int_0^x(x-t)^{k-1}s(t)dt,$$

$$A(x;s(t))=x^{-1}\int_0^\infty e^{-t/x}s(t)dt,$$

the first integral being a Lebesgue integral and the second a Cauchy-Lebesgue integral (the limit as $h\rightarrow\infty$ of the Lebesgue integral from 0 to h). If $0<k\leqq1$, if $s(t)\epsilon S$, and if $s(t)$ is summable C_k to σ [$C_k(x,s(t))\rightarrow\sigma$ as $x\rightarrow\infty$] then $s(t)$ is summable A to σ [$A(x,s(t))$ exists and converges to σ as $x\rightarrow\infty$]. If $k>1$, there exists a function $s(t)\epsilon S$ which is summable C_k and such that for each $x>0$ the Abel transform $A(x,s(t))$ fails to exist. Thus, for summability of functions in class S, the inclusion relation $A\supset C_k$ holds when $0<k\leqq1$ but fails when $k>1$. However, if $k>0$, if $s(t)\epsilon S$, if $s(t)$ is summable C_k to σ, and if the Abel transform $A(x,s(t))$ exists, then $s(t)$ is summable A to σ. *R. P. Agnew.*

Bosanquet, L. S. Note on convergence and summability factors. J. London Math. Soc. 20, 39–48 (1945). [MF 15933]

Conditions for Cesàro summability and for absolute Cesàro summability of series of products are established. Let r and k be integers for which $0\leqq r\leqq k$. The series $\sum\epsilon_na_n$ is summable C_r whenever $\sum a_n$ is summable C_k [or bounded C_k] if and only if

(i) $\epsilon_n=O(n^{r-k})$ [or $o(n^{r-k})$], (ii) $\sum n^k|\Delta^{k+1}\epsilon_n|<\infty$.

The series $\sum\epsilon_na_n$ is summable $|C_r|$ whenever $\sum a_n$ is summable $|C_k|$ if and only if

(iii) $\epsilon_n=O(n^{r-k})$, (iv) $\Delta^k\epsilon_n=O(n^{-k})$.

Let $0\leqq r\leqq k+1$. The series $\sum\epsilon_na_n$ is summable $|C_r|$ whenever $\sum a_n$ is summable C_k if and only if

(v) $\sum n^{k-r}|\epsilon_n|<\infty$, (vi) $\sum n^k|\Delta^{k+1}\epsilon_n|<\infty$.

There are references to several authors who previously obtained parts of these results. Apart from some special cases where $r=0$ or $r=k$, it is not known whether the conditions in (i) to (vi) remain necessary and sufficient when r and k are numbers other than integers. *R. P. Agnew.*

Picone, Mauro. Sul limite del quoziente di due funzionali reali. Boll. Un. Mat. Ital. (2) 5, 120–123 (1943). [MF 16097]

Let $a_0(t)$, $a_1(t)$, $\cdots$ be real nonnegative functions, defined over a set having a limit point t_0, such that

$$\lim_{t\to t_0}a_k(t)=0;\quad\lim_{t\to t_0}\sum_{k=0}^{\infty}a_k(t)=1.$$

It is then well known and easy to show that, for each real bounded sequence s_n,

$$(*)\quad\liminf_{n\to\infty}s_n(t)\leqq\liminf_{t\to t_0}\sigma(t)\leqq\limsup_{t\to t_0}\sigma(t)\leqq\limsup_{n\to\infty}s_n,$$

where $\sigma(t)=a_0(t)s_0+a_1(t)s_1+\cdots$. [For theorems giving necessary as well as sufficient conditions for (*), see W. A. Hurwitz, Proc. London Math. Soc. (2) 26, 231–248 (1927).] Without recourse to the above, (*) is stated and proved for

special cases in which

$$a_k(t)=b_k\varphi_k(t)\Big/\sum_{n=0}^{\infty}b_n\varphi_n(t)>0$$

and $s_k=c_k/b_k$. *R. P. Agnew* (Ithaca, N. Y.).

Agnew, Ralph Palmer. On Hurwitz-Silverman-Hausdorff methods of summability. Tôhoku Math. J. 49, 1–14 (1942). [MF 14693]

Let $H(\chi)$ denote a regular Hurwitz-Silverman-Hausdorff method of summability, that is to say, a transformation of the form

$$\sigma_n=\sum_{k=0}^{n}\left\{\sum_{j=k}^{n}(-1)^{j-k}\binom{n}{j}\binom{j}{k}\mu_j\right\}s_k,$$

where $\mu_n=\mu(n)$ for $n=0, 1, 2, \cdots$ and $\mu(z)=\int_0^1 t^z d\chi(t)$ for $\Re z\geqq 0$, $\chi(t)$ being a complex-valued function of bounded variation over $0\leqq t\leqq 1$ such that $\chi(0+)=\chi(0)=0$ and $\chi(1)=1$. The author's main result is as follows. If q is a complex constant for which $q\neq 0$, $\Re q\geqq 0$, then the sequence $\{s_k{}^q\}\equiv\{\Gamma(k+1)/\Gamma(k-q+1)\}$ is summable $H(\chi)$ if and only if $\mu(q)=0$; if $q\neq 0$, $\Re q\geqq 0$, and $\mu(q)=0$, then the sequence $\{s_k{}^q\}$ is summable $H(\chi)$ to 0. Several consequences of this theorem are established, among which are the following. (i) If $H(\chi_2)\supset H(\chi_1)$ then each zero of $\mu_1(z)$ with nonnegative real part is also a zero of $\mu_2(z)$. As a corollary of (i) it is shown that (ii) if $\mu(z)$ has a zero with nonnegative real part then Abel summability does not include $H(\chi)$. (iii) If $H(\chi)$ permits omission (or adjunction) of elements and if $\Re q\geqq 1$ and $\mu(q)=0$, then $\mu(q-1)=0$. Attention is called to the fact that (iii) implies: (iv) if $H(\chi)$ permits omission (or adjunction) of elements then $H(\chi)$ must have an inverse. In a final section the author considers totally regular transformations $H(\chi)$ and gives some examples. *J. D. Hill.*

Agnew, R. P. Summability of power series. Amer. Math. Monthly 53, 251–259 (1946). [MF 16449]

Let the matrix (b_{nk}) determine a series-to-sequence transformation B by means of which a given series $\sum u_k$ is said to be summable to the value S (or $B\{\sum u_k\}=S$) if each of the series $S_n=\sum_k b_{nk}u_k$, $n=0, 1, \cdots$, converges and $S_n\to S$. The problem arises of finding necessary and sufficient conditions on the matrix (b_{nk}) in order that the transformation B be such that $B\{\sum u_k\}=\sum u_k$ whenever $\sum u_k$ belongs to a specified class K. If K is the class of convergent series or the class of absolutely convergent series the solutions are known. These are stated by the author with historical comments. The author goes on to solve the problem for the case in which K is the class of series $\sum u_k$ such that the power series $\sum u_k z^k$ has a radius of convergence greater than $R\geqq 1$. The proof follows standard lines. As a corollary a theorem of Schur and Szász on power series is shown to follow very simply. *J. D. Hill* (East Lansing, Mich.).

Rajagopal, C. T. On the limits of oscillation of a function and its Cesàro means. Proc. Edinburgh Math. Soc. (2) 7, 162–167 (1946). [MF 16212]

Let

$$\sigma_p(x)=\frac{p}{x}\int_0^x (x-t)^{p-1}s(t)dt,\quad p=1, 2, \cdots;\ \sigma_0(x)=s(x),$$

$$\bar{\sigma}_p=\lim_{x\to\infty}\ {}^{\sup}_{\inf}\ \sigma_p(x),$$

$$w(\lambda)=\limsup_{t\to\infty}\ \max_{t\leqq t'\leqq\lambda t}\ |s(t')-s(t)|.$$

The author gives inequalities in which $\bar{\sigma}_0$ and $\underline{\sigma}_0$ are governed by $\bar{\sigma}_p$, $\underline{\sigma}_p$ and $w(\lambda)$. The results cover various known Tauberian theorems. *L. S. Bosanquet* (London).

Chen, Kien-Kwong. Some one-sided Tauberian theorems. Anais Acad. Brasil. Ci. 17, 249–259 (1945). [MF 15537]

The author modifies an argument of Karamata to prove that, if $\sum a_n$ is summable (A) and $\alpha\geqq -1$, then it is summable $(C, \alpha+1)$ if either (i) the Cesàro sums of order α of $\sum a_n$ are bounded below or (ii) the Cesàro sums of order α of $\sum na_n$ exceed $-cn$ for some constant n. He deduces that, if $\sum a_n$ is summable (A) and $a_n-a_{n-1}<c/n^2$, then $na_n=o(1)$ and proves that the Cesàro sums of order $\alpha+1$ of $\sum a_n$ are bounded if condition (i) is satisfied and if $\sum a_n x^n$ is merely bounded above. The remaining theorems (1 and 4) in the paper are false, since they imply that Abel and Cesàro sums of a series with bounded partial sums necessarily have the same upper and lower limits. The error comes in assuming that the terms in the sum on the right of the first inequality of theorem 4 are all positive. *H. R. Pitt* (Belfast).

Dufresnoy, Jacques. Extension de deux théorèmes de M. Fejèr. C. R. Acad. Sci. Paris 222, 945–946 (1946). [MF 16319]

If the Abel limit of $\sum a_n$ is s, and if $\sum n|a_n|^2<\infty$, then $\sum a_n=s$. This main result of the author and the simple proof are well known [see, for instance, E. Landau, Darstellung und Begründung einiger neuerer Ergebnisse der Funktionentheorie, Springer, Berlin, 1916].
W. W. Rogosinski (Newcastle-upon-Tyne).

Hardy, G. H., and Littlewood, J. E. Notes on the theory of series. XXIV. A curious power-series. Proc. Cambridge Philos. Soc. 42, 85–90 (1946). [MF 15664]

[Note XXIII appeared in the same Proc. 40, 103–107 (1944); these Rev. 6, 47.] Answering a question put by W. R. Dean, the authors prove first that, θ being any irrational number, if the radius of convergence of the series (S) $\sum x^n/\sin n\pi\theta$ is ρ, then the radius of convergence of the series (T) $\sum x^n/(\sin \pi\theta \sin 2\pi\theta \cdots \sin n\pi\theta)$ is $\rho/2$. The proof is based on the use of an algebraic identity. Since for almost all θ the radius ρ is equal to 1, it follows that, for almost all θ, the radius of convergence of (T) is $\frac{1}{2}$. From this the authors deduce that for almost all θ

$$(*)\qquad \lim_{n\to\infty} n^{-1}\sum_{m=1}^{n}\log|\operatorname{cosec} m\pi\theta|=\log 2.$$

They observe that if (*) is true the result about the radii of convergence of (S) and (T) may be extended to the more general series $\sum a_n x^n/\sin n\pi\theta$ and

$$\sum a_n x^n/(\sin \pi\theta \sin 2\pi\theta \cdots \sin n\pi\theta).$$

The authors then give a direct proof of the equality (*). Indeed, they prove more generally that, if $f(x)$ has period 1, is Riemann integrable in $(\delta, 1-\delta)$ for every $\delta>0$, increases steadily to infinity when $x\to 0$ and $x\to 1$, and if

$$\int_0^1 f(x)\{\log^2(1/x)+\log^2(1/(1-x))\}dx<\infty,$$

then the following extension of Weyl's classical theorem is true for almost all θ:

$$\lim_{n\to\infty} n^{-1}\sum_{m=1}^{n} f(m\theta)=\int_0^1 f(x)dx.$$

Finally, they show that the last equality is true whenever $\int_0^1 f(x)dx < \infty$ if the partial quotients of the expansion of θ in a continued fraction are bounded. *R. Salem.*

Selmer, Ernst S. Infinite products for π. Norsk Mat. Tidsskr. **28**, 20 (1946). (Norwegian) [MF 16693]

Rosolini, Amleto. Sulla definizione di valore generalizzato per i determinanti infiniti. Boll. Un. Mat. Ital. (2) **5**, 95–102 (1943). [MF 16093]

A method, analogous to the Abel method for evaluation of series, is proposed for the evaluation of infinite determinants. The infinite determinant

$$\begin{vmatrix} 1+a_{00} & ta_{01} & t^2a_{02} & \cdots \\ ta_{10} & 1+t^2a_{11} & t^3a_{12} & \cdots \\ t^2a_{20} & t^3a_{21} & 1+t^4a_{22} & \cdots \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix}$$

with $t=1$ being given, insert the powers of t and let $\Delta(a_{mn}, t)$ denote the resulting determinant. The coefficient of a_{mn} is t^{m+n}. If, when $0 \leqq t < 1$, this determinant is convergent to $A(a_{mn}, t)$ and is absolutely convergent in the sense defined by Helge von Koch, and if $A(a_{mn}, t) \to A(a_{mn})$ as $t \to 1-$, then the given determinant $\Delta(a_{mn}, 1)$ is evaluable A to $A(a_{mn})$. The method A is regular in the sense that, if $\Delta(a_{mn}, 1)$ converges to D and converges absolutely, then it is evaluable A to D. Two examples are given: $A((-1)^{n+1}) = \frac{1}{2}$; and $A(a_{mn}) = \frac{1}{2}$ in case $a_{mn} = (-1)^{m+n+1}$ when $0 \leqq m \leqq n$ and $a_{mn} = (-1)^{m+n}$ when $0 \leqq n < m$. There are no references to similar work. In particular, there is no comparison with the work of W. L. Hart [Bull. Amer. Math. Soc. **28**, 171–178 (1922)], where, so far as the reviewer knows, such questions were first treated. *R. P. Agnew* (Ithaca, N. Y.).

Schwerdtfeger, H. Moebius transformations and continued fractions. Bull. Amer. Math. Soc. **52**, 307–309 (1946). [MF 16195]

R. E. Lane [same Bull. **51**, 246–250 (1945); these Rev. **6**, 211] has shown how the criterion for convergence of a periodic continued fraction can be derived from simple properties of the Moebius transformation. The author has simplified the derivation of this criterion still further by using the pure transformation point of view and by employing the notions of attractive, repulsive and indifferent fixed points, and has expressed the final theorem in terms of the theory of Moebius transformations only. *H. S. Wall.*

Frank, Evelyn. Corresponding type continued fractions. Amer. J. Math. **68**, 89–108 (1946). [MF 15492]

The author develops a new method of obtaining the continued fraction of the form

$$(1) \qquad 1 + \overset{\infty}{\underset{1}{K}}(a_n z^{\alpha_n}/1)$$

which corresponds to an arbitrary power series $1+\sum_1^\infty c_n z^n$. Her method avoids the division of one power series by another. Padé approximants of continued fractions of the form (1) are developed and illustrated graphically. Rational transformations of (1) are given. Application is made to the continued fraction of Ramanujan. *W. Leighton.*

Bissinger, B. H., and Herzog, F. An extension of some previous results on generalized continued fractions. Duke Math. J. **12**, 655–662 (1945). [MF 15508]

Generalizations and considerable improvements of some of the recent results by the authors [Duke Math. J. **12**, 325–334 (1945); these Rev. **7**, 13]. *M. Kac.*

Fourier Series and Generalizations, Integral Transforms

★Vitali, G., e Sansone, G. Moderna Teoria delle Funzioni di Variabile Reale. 2d ed., vol. II. Consiglio Nazionale delle Ricerche. Monografie di Matematica Applicata. Nicola Zanichelli, Bologna, 1946. viii+511 pp.

This volume, by Sansone, is subtitled Sviluppi in Serie di Funzionali Ortogonali. The topics discussed are, by chapters: (1) Developments in orthogonal functions, introduction to Hilbert space; (2) Fourier series; (3) Series of Legendre polynomials and of spherical harmonics; (4) Laguerre and Hermite series; (5) Approximation and interpolation; (6) Stieltjes integrals (including a discussion of Fourier transforms of distribution functions; this chapter has little connection with the rest of the book). The edition of 1935 was considerably shorter. The principal additions are as follows. Chapter 1: a brief discussion of mean-p convergence. Chapter 2: additional convergence and summability theorems. Chapter 3: additional theorems on Legendre polynomials; a discussion of Laplace series and their (C, k) and Abel summability. Chapter 5 is new. It contains several proofs of Weierstrass's approximation theorem in one and two variables, with some results on degree of approximation; Bernstein's and Markoff's theorems on derivatives of polynomials and trigonometric polynomials; a discussion of best approximation and Chebyshev polynomials; a variety of results on the convergence of various interpolating polynomials.

The book would be suitable as a text for a second course in functions of a real variable. The exposition is clear and rigorous, and original in many places. The point of view is modern, the Lebesgue integral being used throughout.

R. P. Boas, Jr. (Providence, R. I.).

Zygmund, A. On the theorem of Fejér-Riesz. Bull. Amer. Math. Soc. **52**, 310–318 (1946). [MF 16196]

Considering partial sums $U^{(n)}(z)$ of the series

$$U(re^{i\theta}) = \sum_0^\infty (a_n \cos n\theta + b_n \sin n\theta) r^n,$$

the author shows that

$$\int_{-1}^1 |U_\rho^{(n)}(\rho)| d\rho \leqq C_n \int_0^{2\pi} |U_\theta(e^{i\theta})| d\theta,$$

where $C_1 = 2/\pi$ and $\lim C_n = C^* > \frac{1}{2}$. The value $\frac{1}{2}$ for C^* might be expected from the inequality of Fejér and Riesz [Math. Z. **11**, 305–314 (1921)] and its extension to harmonic functions [A. Zygmund, Trans. Amer. Math. Soc. **36**, 586–617 (1934), note III, pp. 599–602], but the result actually shows a close analogy with the Gibbs phenomenon and $C^* = (1/\pi)\int_0^\pi x^{-1} \sin x dx$. The problem is reduced to a consideration of $\sum_1^n r^{m-1} \sin mt$ and $\sum_1^n r^{m-1} \sin mt - \frac{1}{2} r^{n-1} \sin nt$. A partial solution of the corresponding problem for analytic functions is deduced. *A. J. Macintyre* (Aberdeen).

Salem, R., and Zygmund, A. Capacity of sets and Fourier series. Trans. Amer. Math. Soc. **59**, 23–41 (1946). [MF 15316]

It is well known that, if $\sum \log n(a_n^2+b_n^2)$ converges, then the Fourier series

$$(1) \qquad \sum (a_n \cos n\theta + b_n \sin n\theta)$$

converges for almost all θ. Sets of measure zero can be further classified according to their capacity and thus the

sets of divergence of Fourier series can be further investigated. A set E which is situated on the unit circle is said to be of positive α-capacity if there exists a nondecreasing function $\mu(t)$ defined for $0 \leqq t \leqq 2\pi$ such that $\int_E d\mu = 1$ for which the integral

$$\int_E |e^{it} - re^{ix}|^{-\alpha} d\mu(t)$$

is uniformly bounded in x as $r \to 1$. There is a connection between the positive α-capacity of a set and its Hausdorff dimension of order α. [See Frostman, Potentiel d'Équilibre et Capacité des Ensembles, Lund, 1935, p. 86.] A set of α-capacity zero is, of course, of Lebesgue measure zero and the smaller the α the "thinner" the set.

The authors give a necessary and sufficient condition in terms of Fourier series for a set to be of positive α-capacity. They prove that, if $\sum n^\alpha(a_n^2 + b_n^2)$ is finite, then the set of points of divergence of the Fourier series (1) is of $(1-\alpha)$-capacity zero if $\alpha < 1$ and of logarithmic capacity zero if $\alpha = 1$. The latter result is due to Beurling [Acta Math. **72**, 1–13 (1939); these Rev. **1**, 226]. They give some theorems on equi-convergence of Fourier series and certain integrals which enable them to deduce that, if $F(x)$ is continuous and nondecreasing in $(0, 2\pi)$ and belongs to Lip α, $0 < \alpha < 1$, then the points of divergence of the integral

$$\int_0^\pi \{F(x+t) - F(x-t)\} t^{-1-\alpha} dt$$

form a set of α-capacity zero, and some other theorems of a similar type. *A. C. Offord* (Newcastle-upon-Tyne).

Salem, R., and Zygmund, A. The approximation by partial sums of Fourier series. Trans. Amer. Math. Soc. **59**, 14–22 (1946). [MF 15315]

Let $f(x)$ be a continuous function of period 2π and $s_n(x)$ the nth partial sum of its Fourier series. It is known that $s_n(x) - f(x) = O(n^{-\alpha} \log n)$, uniformly in x, if f satisfies $f(x+h) - f(x) = O(|h|^\alpha)$, $0 < \alpha \leqq 1$, uniformly in x (f belongs to Lip α). The factor $\log n$ cannot be omitted [A. Zygmund, Trigonometrical Series, Warsaw-Lwów, 1935, p. 61]. The authors investigate whether the additional condition that f is of bounded variation will give a better result. The answer is negative. If, however, f is continuous, belongs to Lip α, $0 < \alpha < 1$, and $f(x) + Cx$ is monotone for a suitable constant C, then $s_n(x) - f(x) = O(n^{-\alpha})$, uniformly in x. The remaining theorems compare the order of $|s_n - f|$ with that of $|\tilde{s}_n - \tilde{f}|$ for the conjugate series. Thus, if f is continuous and $s_n(x) - f(x) = O(n^{-\alpha})$, $\alpha > 0$, uniformly in x, then also $\tilde{s}_n(x) - \tilde{f}(x) = O(n^{-\alpha})$ uniformly in x. *W. W. Rogosinski.*

Zygmund, A. The approximation of functions by typical means of their Fourier series. Duke Math. J. **12**, 695–704 (1945). [MF 15512]

The author considers the problem of approximating to the periodic continuous function

$$f(x) \sim \tfrac{1}{2}a_0 + \sum_{\nu=1}^{\infty} (a_\nu \cos \nu x + b_\nu \sin \nu x)$$

by typical means

$$X_\omega^k(x) = \tfrac{1}{2}a_0 + \sum_{\nu<\omega} (a_\nu \cos \nu x + b_\nu \sin \nu x)(1 - \nu^k/\omega^k)$$

of its Fourier series, k and ω being positive integers. Results are obtained, of which the first runs as follows. If k is even, if the derivative $f^{(h)}(x)$ exists for some $h \leqq k$ and if $|f^{(h)}(x)| \leqq M$ for all x, then $|f(x) - X_n^k(x)| \leqq MA_{h,k} n^{-h}$, where the positive constant $A_{h,k}$ depends only on h and k. If $h \leqq k-1$ and if $f^{(h)}(x)$ has a modulus of continuity $\omega(\delta)$, then $|f(x) - X_n^k(x)| \leqq B_{h,k}\omega(2\pi/n)$. Approximation to the conjugate function is also considered. *E. H. Linfoot.*

✶Nikolsky, S. Approximations of periodic functions by trigonometrical polynomials. Trav. Inst. Math. Stekloff **15**, 76 pp. (1945). (Russian. English summary)

The author investigates the degree of approximation of periodic functions by the partial sums and the $(C, 1)$ means of their Fourier series. The functions are supposed to possess a certain number of derivatives, the last of which, in turn, satisfies conditions like Lip α or boundedness. The following is typical of the results obtained. Let $M = M(r, \alpha, K)$ be the class of all functions of period 2π, such that the rth derivative $f^{(r)}(x)$ exists and satisfies the condition

$$|f^{(r)}(x+h) - f^{(r)}(x)| \leqq K|h|^\alpha$$

for all x and h (r is nonnegative but not necessarily an integer). Let $E_n = \sup |f(x) - S_n(x)|$ for all x and all $f \epsilon M$, S_n being the partial sum of the Fourier series of f. Then

$$E_n = \frac{2^{\alpha+1}K}{\pi^2} \frac{\log n}{n^{r+\alpha}} \int_0^{\pi/2} v^\alpha \sin v dv + O(n^{-r-\alpha}).$$

[See also A. Kolmogoroff, Ann. of Math. (2) **36**, 521–526 (1935).] Similar estimates are obtained for the approximation by $(C, 1)$ means and also for the conjugate functions. Parallel results are given for the approximation by the Lagrange and the Fejér-Bernstein interpolating polynomials. *A. Zygmund* (Philadelphia, Pa.).

Szász, Otto. On the absolute convergence of trigonometric series. Ann. of Math. (2) **47**, 213–220 (1946). [MF 16331]

If (∗) $|a_{n+1}| \leqq c|a_n|$ for fixed $c > 0$, the absolute convergence of $\sum_0^\infty a_n \cos nx$ at one point x_0 implies absolute convergence for all x. The same holds for $\sum_1^\infty a_n \sin nx$ if $x_0 \not\equiv 0 \pmod \pi$. This generalizes a theorem of P. Fatou [Bull. Soc. Math. France **41**, 47–53 (1913)], in which the hypothesis $|a_{n+1}| \leqq |a_n|$ is made instead of (∗). Results of a similar character are obtained for $\sum_0^\infty a_n \cos \lambda_n x$.

A generalization in a slightly different direction is the following result. If $\varphi(x) \not\equiv 0$ is a Riemann integrable function of period one, and if $|a_{n+1}| \leqq (1 + c/n)|a_n|$ for some $c > 0$ and all n, the absolute convergence of $\sum_1^\infty a_n \varphi(nx)$ for some irrational x implies the convergence of $\sum_1^\infty |a_n|$. *P. Civin* (Eugene, Ore.).

Menchoff, D. Sur les séries trigonométriques universelles. C. R. (Doklady) Acad. Sci. URSS (N.S.) **49**, 79–82 (1945). [MF 16399]

We denote by S_n the partial sums of a trigonometric series (S) $\sum(a_n \cos nx + b_n \sin nx)$. It is well known that there exist series S with coefficients tending to 0, such that not only $\{S_n(x)\}$, but also any subsequence $\{S_{n_k}\}$, $n_1 < n_2 < \cdots$, diverges almost everywhere (take, for example, for S the series $\sum \alpha_n \cos 2^n x$, with $\alpha_n \to 0$, $\sum \alpha_n^2 = \infty$ [see the reviewer's "Trigonometrical Series," Warsaw-Lwów, 1935, p. 120]). However, (1) every trigonometric series S, whether convergent or divergent, can be decomposed into two series $S^{(1)}$ and $S^{(2)}$ such that $\{S^{(1)}_{m_k}\}$ and $\{S^{(2)}_{n_k}\}$ converge almost everywhere for suitable $\{m_k\}$ and $\{n_k\}$. A trigonometric series S is called universal if, given any measurable function $f(x)$ of period 2π, we can find a sequence $n_1 < n_2 < \cdots$ such that $S_{n_k} \to f$ almost everywhere. (2) The series $S^{(1)}$ and $S^{(2)}$ in (1) can always be made universal, and their coefficients

can be made $o(1)$, if $a_n, b_n = o(1)$. [The notion of universal trigonometric series is analogous to the notion of universal primitive function; see J. Marcinkiewicz, Fund. Math. 24, 305–308 (1935).] *A. Zygmund* (Philadelphia, Pa.).

Romano, Anna Maria. Sul teorema di Jordan per le serie doppie di Fourier. Ann. Scuola Norm. Super. Pisa (2) 12, 85–97 (1943). [MF 16768]

Let $f(x, y)$ be a function of period 2π with respect to each variable and of bounded variation in Tonelli's sense over the square $0 \leqq x \leqq 2\pi$, $0 \leqq y \leqq 2\pi$. It is well known that the double Fourier series of f converges almost everywhere [Tonelli, same Ann. (2) 6, 315–326 (1935); L. Cesari, Rend. Sem. Mat. Univ. Roma (4) 1, 277–294 (1937)]. It is now shown by the author that this result cannot be improved upon even if $f(x, y)$ is, in addition, continuous. More precisely, the set of points of divergence of the Fourier series of such an f can actually be of the power of the continuum in every square. Also the Fourier series can converge everywhere but uniformly in no square. *A. Zygmund.*

Čelidze, V. G. The Abel-Poisson method of summation of double Fourier series. Trav. Inst. Math. Tbilissi [Trudy Tbiliss. Mat. Inst.] 13, 79–99 (1944). (Russian. Georgian summary) [MF 14621]

It is proved that the double Fourier series of a Lebesgue integrable function $f(x, y)$ is Abel summable almost everywhere. This theorem was first established by Marcinkiewicz and Zygmund [Fund. Math. 32, 122–132 (1939)]. The statement and proof of lemma 1 of the present paper are almost identical with those given by Marcinkiewicz and Zygmund. *M. Kac* (Ithaca, N. Y.).

Civin, Paul. Fourier coefficients of dominant functions. Duke Math. J. 13, 1–7 (1946). [MF 15869]

Let $f(x) \sim \sum_{-\infty}^{\infty} c_n e^{inx}$ and $F(x) = \sum_{-\infty}^{\infty} C_n e^{inx}$ be two functions of period 2π such that $|f(x)| \leqq F(x)$. Denote the sums $(\sum_{-\infty}^{\infty} |c_n|^r)^{1/r}$ and $(\sum_{-\infty}^{\infty} |C_n|^r)^{1/r}$ by $N_r(f)$ and $N_r(F)$, respectively. The author proves that $N_q(f) \leqq N_q(F)$, $q = 2, 4, 6, \cdots$, and that corresponding to any given $f(x)$ we can find a dominant function $F_p(x)$ such that $N_p(F_p) \leqq N_p(f)$ for $p = 2k/(2k-1)$, $k = 1, 2, \cdots$. *A. C. Offord.*

Civin, Paul. Polynomial dominants. Bull. Amer. Math. Soc. 52, 352–356 (1946). [MF 16205]

Suppose that $f(x)$ is a continuous function of period 2π and that $F_n(x)$ is a trigonometric polynomial such that $|f(x_j^n)| \leqq F_n(x_j^n)$, where $x_j^n = 2j\pi/(2n+1)$, $j = 0, 1, \cdots, 2n$. The author proves that, for $1 < r < \infty$,

$$\left\{\int_0^{2\pi} |f(x)|^r dx\right\}^{1/r} \leqq A_r \left\{\int_0^{2\pi} |F_n(x)|^r dx\right\}^{1/r} + B_r \omega(2\pi/n),$$

where A_r and B_r are numbers depending on r alone and $\omega(\delta)$ is the modulus of continuity of $f(x)$. He also gives the corresponding results for $0 < r < 1$ and for $r = 1$. The proofs employ the methods of Marcinkiewicz and Zygmund [Fund. Math. 28, 131–166 (1937)]. *A. C. Offord.*

Favard, J. Sur les multiplicateurs d'interpolation. J. Math. Pures Appl. (9) 23, 219–247 (1944). [MF 15741]

Functions $A_n^k(x)$, $k = 0, 1, \cdots, n-1$; $n = 1, 2, \cdots$, defined for $0 \leqq x \leqq 1$, are called multiplicateurs d'interpolation provided that we have

$$(1) \qquad \lim_{n\to\infty} \sum_{k=0}^{n-1} f(k/n) A_n^k(x) = f(x),$$

in the range $0 < x < 1$, for functions $f(x)$ $(0 \leqq x \leqq 1)$ satisfying suitable conditions. The author derives (i) sufficient conditions for nonnegative $A_n^k(x)$ in order that (1) hold for $f(x)$ continuous in $0 \leqq x \leqq 1$, (ii) sufficient conditions for $A_n^k(x)$, of arbitrary sign, in order that (1) hold for $f(x)$ continuous and of bounded variation. [In this connection H. Hahn's paper [Math. Z. 1, 115–142 (1918)] should be mentioned.] Beyond these results, which are also extended to the case of two variables, a large number of sets of multiplicateurs d'interpolation are obtained from various classical singular integrals of probability theory and analysis, by replacing these integrals by suitable approximating sums. We mention two examples related to Weierstrass's singular integral. The two sums

$$(n/\pi)^{\frac{1}{2}} \sum_{k=-\infty}^{\infty} f(k/n) \int_{k/n}^{(k+1)/n} \exp\{-n(t-x)^2\} dt,$$

$$(n\pi)^{-\frac{1}{2}} \sum_{k=-\infty}^{\infty} f(k/n) \exp\{-n(k/n-x)^2\}$$

are shown to converge to $f(x)$ uniformly in every finite interval, provided that $f(x)$ is continuous and suitably restricted in growth in order to insure convergence. *I. J. Schoenberg* (Philadelphia, Pa.).

Kac, M. On the distribution of values of sums of the type $\sum f(2^k t)$. Ann. of Math. (2) 47, 33–49 (1946). [MF 15657]

(i) Let $f(t)$ be a function of period 1, satisfying Hölder's condition of order greater than $\frac{1}{2}$. Suppose, moreover, that

$$n^{-1} \int_0^1 \left\{\sum_0^n f(2^k t)\right\}^2 dt \to \sigma^2 \neq 0.$$

Then the measure of the set of t's for which

$$a < n^{-\frac{1}{2}} \sum_0^n f(2^k t) < b$$

approaches, as $n \to \infty$, the integral

$$(*) \qquad (2\pi)^{-\frac{1}{2}} \sigma^{-1} \int_a^b \exp(-u^2/2\sigma^2) du.$$

(ii) If $f(t)$ satisfies the same conditions, and if $l_{k+1} - l_k$ is increasing, the measure of the set of points where

$$a < n^{-\frac{1}{2}} \sum_1^n f(2^{l_k} t) < b$$

approaches, as $n \to \infty$, the integral (*), where $\sigma^2 = \int_0^1 f^2 dt$. (iii) If $l_{n+1} - l_n$ is increasing and $f(t)$ satisfies Hölder's condition of order greater than $\frac{1}{2}$, the divergence of the series $\sum c_n^2$ implies the divergence almost everywhere of $\sum c_n f(2^{l_n} t)$. [For (i) and (ii) see also Kac, Studia Math. 7, 96–100 (1938); Fortet, Studia Math. 9, 54–70 (1940); these Rev. 3, 169.] *A. Zygmund* (Philadelphia, Pa.).

Foà, Alberto. Sulla sommabilità forte delle serie di Legendre. Boll. Un. Mat. Ital. (2) 5, 18–27 (1943). [MF 16081]

Let $s_n(x)$ be the partial sums of the Fourier-Legendre series of an $f(x) \in L^r(-1, 1)$, $r > 1$. It is shown that, for every $k > 0$ and at every point x inside $(-1, 1)$ at which

$$\int_0^h |f(x \pm t) - f(x)|^r dt = o(h)$$

(in particular, almost everywhere), we have

$$(n+1)^{-1}\sum_{\nu=0}^{n}|s_\nu(x)-f(x)|^k\to 0$$

as $n\to\infty$. This is an analogue of the well-known result of Hardy and Littlewood [see, for example, the reviewer's Trigonometrical Series, Warsaw-Lwów, 1935, p. 238].

A. Zygmund (Philadelphia, Pa.).

Dalzell, D. P. On the completeness of a series of normal orthogonal functions. J. London Math. Soc. 20, 87–93 (1945). [MF 16700]

Let $\{\varphi_n(x)\}_{n=1}^{\infty}$ be orthonormal on (a, b). It is shown that $\{\varphi_n\}$ is complete in $L^2(a, b)$ if and only if $\sum_{n=1}^{\infty}A_n=1$, where

$$A_n=2(b-a)^{-2}\int_a^b\left[\int_a^t\varphi_n(x)dx\right]^2dt,$$

or, more generally,

$$(b-a)A_n=\int_a^b\left[\int_a^t f(x)\varphi_n(x)dx-\int_t^b f(x)\varphi_n(x)dx\right]^2dt,$$

where $f(x)$ is continuous, $f(x)\neq 0$ i.. $a<x<b$, and

$$\int_a^b f(x)^2dx=1.$$

Applications to specific cases include the Jacobi functions $(1-x)^{\frac{1}{2}\alpha}(1+x)^{\frac{1}{2}\beta}P_n^{\alpha,\beta}(x)$. *R. P. Boas, Jr.*

Dalzell, D. P. On the completeness of Dini's series. J. London Math. Soc. 20, 213–218 (1945).

A modification of the author's completeness criterion [see the preceding review] is applied to prove the (known) result that the set $\{x^{\frac{1}{2}}J_\nu(k_nx)\}_{n=1}^{\infty}$, where the k_n are the positive roots of

$$\lambda^{-\nu}\{\lambda J_\nu'(\lambda)+HJ_\nu(\lambda)\}=0,\qquad \nu>-1,$$

is complete in $L^2(0, 1)$ if and only if $H+\nu>0$, while the set is completed by the addition of one function if $H+\nu\leqq 0$.

R. P. Boas, Jr. (Providence, R. I.).

Sicardi, Francesco. Sulla convergenza in media di talune serie di funzioni reali di quadrato sommabile. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. 76, 531–540 (1941). [MF 16260]

The author considers two sequences $\{\varphi_n(x)\}$, $\{\psi_n(x)\}$ of functions defined on a measurable set T in n-space. The set $\{\varphi_n\}$ is orthonormal and complete in $L^2(T)$; the set $\{\psi_n\}$ satisfies

$$\sum_{n=1}^{\infty}\int_T\{\varphi_n(x)-\psi_n(x)\}^2dx<\infty.$$

Furthermore, each ψ_n is orthogonal to $\varphi_1, \varphi_2, \cdots, \varphi_{n-1}$ and $\int\varphi_n\psi_ndx=1$. Every $f\epsilon L^2(T)$ then has a formal development $f(x)\sim\sum c_n\psi_n(x)$, where the c_n are determined recursively from

$$\int_T f(x)\varphi_m(x)dx=\sum_{n=1}^{m}c_n\int_T\varphi_m(x)\psi_n(x)dx.$$

The author proves that the formal development converges to f on T in mean square. *R. P. Boas, Jr.*

Schwartz, Laurent. Approximation d'une fonction quelconque par des sommes d'exponentielles imaginaires. Ann. Fac. Sci. Univ. Toulouse (4) 6, 111–176 (1943). [MF 15175]

In his thesis the author studied the problem of approximation by linear combinations of real exponential functions [Étude des sommes d'exponentielles réelles, Actualités Sci. Ind., no. 959, Hermann, Paris, 1943; these Rev. 7, 294]. The present sequel is concerned with the corresponding problems for complex exponential functions. The first section is a preliminary survey of entire functions of exponential type and of finite trigonometric integrals; most of the results can be found elsewhere [cf. N. Levinson, Gap and Density Theorems, Amer. Math. Soc. Colloquium Publ., v. 26, New York, 1940; these Rev. 2, 180].

Section two is devoted to a study of sets of functions $e^{-2i\pi\lambda_n y}$, the λ_n real, with respect to the properties of totality and minimality (liberté) in the spaces $L^p(-A, A)$ and $C(-A, A)$. The fundamental property is this. If the set $\{e^{-2i\pi\lambda_n y}\}_{-\infty}^{\infty}$ is not total in L^p, then it is necessarily minimal; if it is total, then it is either minimal (that is, no element is contained in the closed span of the others) or every element is contained in the closed span of the others. It is shown that a set of complex exponentials which is not total, and is of finite deficiency $n>0$, becomes both total and minimal on adjoining n such elements distinct from the others; there is a dual result on the removal of n elements from a system with excess. The difficult problems of necessary and sufficient conditions for totality and minimality are left open.

In the next two sections it is assumed that the set $S:\{e^{-2i\pi\lambda_n y}\}_{-\infty}^{\infty}$ is minimal in one of the spaces L^p. The importance of minimality is that it guarantees that S admits a biorthogonal set, so that every element $f(y)$ in the closed span B_A^p of S admits a unique formal development $f(y)\sim\sum_n c_ne^{-2i\pi\lambda_n y}$ (B_A^p is the whole space if and only if S is total, a hypothesis which is not assumed). In the case of ordinary Fourier series the development of $f(y)$ always converges to it in the topology of the space. For the general Fourier series considered here mean-convergence is an open question [except possibly in closed proper subintervals; cf. Levinson, op. cit., p. 48] and another kind of convergence is considered instead. It is proved that, after a grouping of the terms (the grouping depending only on the λ_n), the series $\sum c_ne^{-2i\pi\lambda_n y}e^{-2\pi|\lambda_n|x}$ converges for $x>0$ and all y. The sum $f(y; x)$ is harmonic in the half-plane $x>0$ and for fixed x converges to $f(y)$ in $L^p(-A+\delta, A-\delta)$.

Section five contains applications to the growth properties and Julia lines of Dirichlet series and to the uniqueness of trigonometric series. For example, it is shown that, if $e^{-2\pi\Omega_n y}$ is minimal in $C(-L, L)$, then a series $\sum c_ne^{-2\pi\Omega_n y}$ cannot be summed uniformly to zero in an interval of length greater than or equal to $2L$, by any regular summability method, unless all the c_n vanish.

Preparatory to forthcoming investigations of analytic functions, the paper concludes with a study of the completeness of sets in compact sets of the complex plane.

H. Pollard (Ithaca, N. Y.).

Filonenko-Borodich, M. M. On a certain system of functions and its application in the theory of elasticity. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] 10, 193–208 (1946). (Russian. English summary) [MF 16844]

The system is $P_n(x)=2\sin(\pi x/l)\sin(n\pi x/l)$, $n=1, 2, \cdots$; it is "almost orthogonal" in the sense that $\int_0^lP_n(x)P_m(x)dx=0$ unless $|n-m|=0$ or 2. Hence the coefficients in the formal expansion of any integrable function in terms of the $P_n(x)$ can easily be computed. The author shows that the expansion represents any function satisfying "Dirichlet's conditions" and vanishing at 0 and l. [The author also writes $P_n(x)$ as a difference of cosines; in this form his work suggested the investigation by Stepanoff reviewed below.]

Since the $P_n(x)$ vanish together with their derivatives at 0 and l, they are appropriate for solving the differential equation $EIy^{iv}=q(x)$ with these boundary conditions, that is, the equation for the bending of a beam with clamped ends, and also for the partial differential equation of a rectangular plate with given stresses on its ends. These applications are discussed in detail. *R. P. Boas, Jr.*

Stepanoff, V. Sur quelques systèmes non orthogonaux. C. R. (Doklady) Acad. Sci. URSS (N.S.) 48, 385–388 (1945). [MF 16651]

Let $\{\varphi_n\}$ be an orthonormal set of functions and consider a system $\{\psi_n\}$ of linear combinations $a_n^{(1)}\varphi_1+\cdots+a_n^{(k_n)}\varphi_{k_n}$, subject to the conditions that $a_n^{(k_n)}\neq 0$, $\{k_n\}$ is nondecreasing, $\infty>\liminf (k_n-n)=m\geqq 0$ and the ψ_n are linearly independent. The author discusses the completeness in L^2 of the set $\{\psi_n\}$. He replaces the set $\{\psi_n\}$ by an equivalent set, in terms of which he defines a sequence of determinants by means of which he obtains a necessary and sufficient condition for completeness. The detailed formulation is too lengthy to reproduce here. *R. P. Boas, Jr.*

Kober, H. A note on approximation by rational functions. Proc. Edinburgh Math. Soc. (2) 7, 123–133 (1946). [MF 16207]

Let H_p and $\mathfrak{H}_p$ denote the spaces of regular functions which satisfy, respectively, the conditions

$$\int_{-\pi}^{\pi}|f(re^{i\theta})|^p d\theta<M^p, \qquad 0<r<1,$$

$$\int_{-\infty}^{\infty}|f(x+iy)|^p dx<M^p, \qquad y>0,$$

and H_p', $\mathfrak{H}_p'$ the corresponding spaces of boundary functions. Approximation theorems such as the following are proved. (1) The divergence of the product $\prod|b_n|$ is necessary and sufficient for the sequence $(e^{i\theta}-b_n)^{-1}$ to be closed in H_p', where $1\leqq p<\infty$. (2) The sequence $(x-\beta_n)^{-1}$ is closed in $\mathfrak{H}_p'$ $(1<p<\infty)$ if and only if $\sum\mathfrak{I}\{\beta_n\}/(1+|\beta_n|^2)$ diverges.

It is proved also that the orthogonal sequence

$$\{(a-t)^n(\bar{a}-t)^{-n-1}\},$$

$\mathfrak{I}\{a\}>0$, $n=0, \pm 1, \cdots$, has the following property: if $f(x)$ belongs to $L^p(-\infty, \infty)$ then its development in terms of these functions converges to $f(x)$ in the pth mean. [This last result was proved independently by Caton and Hille using the same method of proof. Cf. Duke Math. J. 12, 217–242 (1945); these Rev. 6, 268.] *H. Pollard.*

Aitken, A. C. A note on inverse central factorial series. Proc. Edinburgh Math. Soc. (2) 7, 168–170 (1946). [MF 16213]

Let $x^{[k-1]}=x^{[k]-1}$, where $x^{[k]}$ $(k\geqq 1)$ is the central factorial in Steffensen's notation. Let also, for $k\leqq -1$, $x^{[k]}=1/x^{[-k]}$. These $x^{[k]}$ $(k\leqq -1)$ are the inverse central factorials which may also be expressed by the formula

$$x^{[-n-1]}=(-1)^n\delta^n(1/x)/n! \qquad n\geqq 0.$$

The relation (1) $\delta x^{[k]}=kx^{[k-1]}$ is found to hold for all integral $k\neq 0$. Starting from the formal expansion

$$1/(x^2-a^2)=x^{[-2]}+a^{[2]}x^{[-4]}+a^{[4]}x^{[-6]}+\cdots,$$

by summation, using (1), the following equally formal expansion

$$\sum_{s=n+a}^{\infty}\frac{1}{x^2-a^2}=\nu^{[-1]}+\tfrac{1}{3}a^{[2]}\nu^{[-3]}+\tfrac{1}{5}a^{[4]}\nu^{[-5]}+\cdots,$$

where $\nu=n+a-\frac{1}{2}$, is obtained and it is shown how it can be used to give very close approximations to the sum of the series $\sum_1^\infty n^{-2}$. Further such examples are given. The main purpose of the note, however, is to point out the desideratum of a theory for the expansion of functions in series of inverse central factorials (2) $a_0x^{[-1]}+a_1x^{[-2]}+a_2x^{[-3]}+\cdots$. It concludes with the following suggestive remark. Since

$$x^{[-n-1]}=(1/n!)\int_0^\infty e^{-xt}(2\sinh t/2)^n dt,$$

an expansion $g(t)=\sum_{n=0}^\infty a_n(2\sinh t/2)^n/n!$ implies for the Laplace transform of $g(t)$ the expansion (2).

I. J. Schoenberg (Philadelphia, Pa.).

Jessen, Børge, and Tornehave, Hans. Mean motions and zeros of almost periodic functions. Acta Math. 77, 137–279 (1945). [MF 14634]

In this paper the authors make a very detailed investigation of the Jensen functions, mean motions and zeros of analytic a.p. (almost periodic) functions. The paper includes a historical introduction and two chapters which are primarily expository, followed by five chapters in which new results are presented. The third chapter contains new results on the distribution of values of a.p. sequences and the fourth deal with analytic a.p. functions connected with a.p. sequences. The fifth chapter shows the existence of analytic a.p. functions which for a given vertical line have upper and lower left and right mean motions and left and right derivatives of the Jensen function with arbitrary preassigned values (satisfying certain necessary inequalities). In chapter VI, necessary and sufficient conditions are given that a given function be the Jensen function for some a.p. function (whose module may or may not be restricted). In the last chapter stronger results are obtained for functions with integral bases and analytic spatial extensions. For instance, mean motions exist for every vertical line in the strip for such functions. *R. H. Cameron.*

Bohr, Harald. Contribution to the theory of analytic almost periodic functions. On the behaviour of an analytic almost periodic function in the neighbourhood of a boundary for its almost periodicity. Danske Vid. Selsk. Math.-Fys. Medd. 20, no. 18, 37 pp. (1943). [MF 15397]

Il s'agit de fonctions $f(s)$ analytiques, presque périodiques et non bornées dans $\{\alpha, \beta)$, c'est-à-dire, analytiques et presque périodiques pour $\alpha<\sigma<\beta$ et continues pour $\alpha\leqq\sigma<\beta$ $(s=\sigma+it)$. On appelle nombre de translation τ d'une telle fonction un nombre réel tel que la fonction $f(s+i\tau)-f(s)$ reste bornée pour $\alpha\leqq\sigma<\beta$; leur ensemble forme un module F. Dans le cas où $\beta=\infty$, F est discret et on montre que toute fonction de cette sorte est la somme d'une fonction périodique $p(s)$ non bornée et d'une fonction $b(s)$ bornée et presque périodique, toutes deux dans $\{\alpha, \infty)$; en un certain sens il y a unicité d'une telle décomposition. Dans le cas où β est fini, F ne peut pas comprendre tous les nombres réels. Si F est discret, on a un résultat tout à fait analogue au précédent; si F est dense partout il y a des fonctions pour lesquelles une telle décomposition n'est pas possible; des exemples en sont donnés au moyen de séries de Dirichlet dont les exposants croissent suffisamment vite.

J. Favard (Paris).

Bohr, Harald. Über das Koeffizientendarstellungsproblem Dirichletscher Reihen. Danske Vid. Selsk. Math.-Fys. Medd. 20, no. 2, 12 pp. (1942). [MF 15398]

Perron has proved [J. Reine Angew. Math. 134, 95–143

(1908)] that from $f(s)=\sum a_n/n^s$ it follows for α greater than the convergence abscissa of the series and for $\rho\geqq 1$ that

$$(1/2\pi i)\int_{a-i\infty}^{a+i\infty} w^s s^{-\rho} f(s)ds=(1/\Gamma(\rho))\sum_{n<w} a_n(\log(w/n))^{\rho-1}.$$

The author shows that for $\rho<1$ this formula is, in general, no longer true. He constructs a series $f(s)$ for which the formula is not valid for any ρ in $0<\rho<1$.

František Wolf (Berkeley, Calif.).

Wuytack, F. Generalization of the Heaviside symbolic calculus. Wis- en Natuurk. Tijdschr. 10, 141–154 (1941). (Dutch) [MF 15578]

Let D be the symbol of differentiation with respect to t, I the symbol of integration from t_0 to t. The author investigates matrix operators

$$T=\sum_1^\infty A_iI^i+\sum_0^k B_jD^j,$$

where A_i and B_j are constant $n\times n$ matrices. He defines $T\cdot f$ and $f\cdot T$ for an $n\times n$ square matrix $f(t)$ whose elements are functions of t with bounded kth derivatives in the interval under consideration. The properties of the operator T depend on the convergence properties of the associated series

$$G_{\alpha\beta}(R)=\sum_1^\infty |a_{\alpha\beta,i}|R^{i-1}/(i-1)!,\qquad A_i=[a_{\alpha\beta,i}].$$

The inverse operator is obtained by inversion of the series

$$T(p)=\sum_1^\infty A_ip^{-i}+\sum_0^k B_jp^j.$$

An alternative representation of $T\cdot f$ is

$$T\cdot f=\int_{t_0}^t h(t-u)\cdot f(u)du+\sum_0^k B_jd^jf(t)/dt^j,$$

$$h(u)=\sum_1^\infty A_iu^{i-1}/(i-1)!.$$

Clearly, $h(u)$ is an analytic function of u if the associated series are convergent.

The results can be used for the solution of integrodifferential equations of the form $T\cdot f=g$. *A. Erdélyi.*

Wuytack, F. Generalization of the symbolic calculus. II. Wis- en Natuurk. Tijdschr. 11, 1–19 (1942). (Dutch) [MF 15580]

[Cf. the preceding review for notations.] In this part the author considers in the first instance operators $H=\sum_1^\infty A_iI^i$ for which the function $h(u)$ is analytic. He investigates power series $\sum_0^\infty C_jH^j$ in such operators and rearranges these series in powers of I. The commutativity of two operators is discussed, and the inversion of A_0+H is obtained as a Neumann series $\sum_0^\infty(-A_0^{-1}H)^nA_0^{-1}$. The investigation is extended to operators H for which the function is not necessarily analytic. The uniqueness of the inversion is discussed.

A. Erdélyi (Edinburgh).

Wuytack, F. Generalization of the symbolic calculus. III. Wis- en Natuurk. Tijdschr. 11, 105–117 (1943). (Dutch) [MF 15585]

The results of parts I and II [cf. the preceding two reviews] are used to generalise the (Heaviside) symbolic calculus. Given an operator $B=\sum_0^kB_jD^j$, P is defined as a root of the algebraic equation $\sum_0^kB_j(-P)^j=p$, and then the transformation of $f(t)$ into

$$F\{f\}=-Pe^{-Pt_0}\int_{t_0}^\infty e^{Pt}f(t)dt$$

is considered. Clearly F is a linear transformation which transforms the unit function into unity and, if $f(t)$ is k times differentiable and if this function together with its first $k-1$ derivatives vanishes for $t=t_0$, then the "F-transform" of Bf is equal to p times the "F-transform" of f itself. This "generalised" Laplace transformation is used for the solution of integrodifferential equations. The inverse transformation is also investigated. *A. Erdélyi* (Edinburgh).

Humbert, Pierre. Nouvelles correspondances symboliques. Bull. Sci. Math. (2) 69, 121–129 (1945). [MF 15888]

Let $\varphi(p)\subset f(t)$ symbolize the relation $\varphi(p)=p\int_0^\infty e^{-pt}f(t)dt$. Suppose that, for some fixed positive a, $p^ae^{-p^a}\subset g_a(t)$. Then, formally at least,

$$\varphi(p^a)\subset\int_0^\infty u^{-1}f(u)g_a(tu^{-1/a})du.$$

In order to make use of this formula it is necessary to compute the function $g_a(t)$ explicitly. The author indicates a method for carrying out the computation. The details are given for special rational values of a. The converse problem of the relation between $\varphi(p)$ and $f(t^{1/a})$ is also discussed, together with other questions of a similar character. All the results are derived formally, but the reviewer finds that a simple rigorous proof can be obtained from the work of Post [Trans. Amer. Math. Soc. **32**, 723–781 (1930), in particular, pp. 730, 733]. *H. Pollard* (Ithaca, N. Y.).

Obreschkoff, Nikola. Sopra gli sviluppi asintotici e la trasformazione di Laplace. Ann. Mat. Pura Appl. (4) 20, 137–140 (1941). [MF 16599]

If $f(s)$ is the Laplace transform of $F(x)$ and has an asymptotic series $\sum a_ns^{-\lambda_n}$, then, formally, $F(x)$ has the asymptotic series $\sum\{a_n/\Gamma(\lambda_n)\}x^{\lambda_n-1}$. The author gives conditions under which this formal relationship can be justified.

R. P. Boas, Jr. (Providence, R. I.).

Amerio, Luigi. Alcuni teoremi tauberiani per la trasformazione di Laplace. Ann. Mat. Pura Appl. (4) 20, 159–193 (1941). [MF 16600]

The author's object is to infer that $F(t)$ is bounded or approaches a limit as $t\to\infty$ from properties of its Laplace transform $f(z)$. He assumes that $F(t)$ is integrable in every finite interval and that $f(z)$ has a half-plane of convergence and is analytic in $x>0$ $(z=x+iy)$. Since $F(t)$ is formally the Fourier transform of the boundary function $f(iy)$, one would expect the properties of $f(iy)$ to be relevant. The author imposes, in the first part of the paper, rather lengthy hypotheses on $f(iy)$ and on the rate of approach of $f(z)$ to its boundary values so that the formal Fourier inversion is actually summable, and infers criteria of the desired form. In the second part, he considers the case where $f(z)$ is of the form $\sum_{n=-\infty}^\infty a_n/(z-i\lambda_n)$ and investigates the structure of $F(t)$. *R. P. Boas, Jr.* (Providence, R. I.).

Akhiezer, N. On some inversion formulae for singular integrals. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 9, 275–290 (1945). (Russian. English summary) [MF 15358]

Let E be an open set. The nonnegative function $p(x)$

belongs to the class A if

$$\tilde{p}(x)=\pi^{-1}\int_E (x-t)^{-1}p(t)dt$$

exists almost everywhere on E and is equal to

$$\alpha+\mu x-\sum_k (x-\alpha_k)^{-1}\mu_k,$$

where $\mu\geqq 0$, $\mu_k>0$ and α_k are real numbers and the α's belong to the complement of the set E and are such that each interval complementary to E contains only a finite number of α's. The author shows that, if $p(x)$ and $q(x)=1/p(x)$ both belong to A and if $\tilde{q}(x)=\beta+\nu x-\sum_k(x-\beta_k)^{-1}\nu_k$, then the sequences $S_\alpha\{\alpha_k\}$ and $S_\beta\{\beta_k\}$ have no point in common.

If, furthermore, $H\{p(x)\}$ denotes the set of functions $f(x)$ which are such that $\int_E f^2(x)p(x)dx<\infty$ and

$$\int_E (x-c)^{-1}f(x)p(x)dx=0$$

for every c belonging to S_β and if $H\{q(x)\}$ is defined similarly, then, subject to a certain restriction on E, if $g(x)$ is any function of $H\{q(x)\}$ the equation

$$g(x)=\pi^{-1}\int_E (x-t)^{-1}f(t)p(t)dt$$

has a unique solution $f(x)$ belonging to $H\{p(x)\}$, given by the formula obtained by interchanging f and g in this equation. Furthermore,

$$\int_E p(x)|f(x)|^2dx=\int_E q(x)|g(x)|^2dx.$$

A. C. Offord (Newcastle-upon-Tyne).

Polynomials, Polynomial Approximations

Visser, C. A simple proof of certain inequalities concerning polynomials. Nederl. Akad. Wetensch., Proc. 48, 276–281 = Indagationes Math. 7, 81–86 (1945). [MF 15800]

The author gives elegant and very simple proofs of two known results. Namely, if $P(x)$ is a polynomial of degree n and the coefficient of x^n is 1, then

$$\max_{-1\leqq x\leqq 1}|P(x)|\geqq 2^{-n+1},\qquad \int_{-1}^{1}|P(x)|dx\geqq 2^{-n+1}.$$

There is equality if and only if $P(x)$ is $2^{-n+1}\cos nt$ or $2^{-n}\sin(n+1)t\operatorname{cosec} t$, respectively, where $\cot t=x$.

A. C. Offord (Newcastle-upon-Tyne).

Germain, Paul. Étude de l'approximation de certaines fonctions à l'aide de polynômes. Revue Sci. (Rev. Rose Illus.) 82, 298–303 (1944). [MF 16474]

The author discusses three problems. (I) Find a polynomial $P_n(x)$ of degree n so that $P_n(x)\geqq 0$ in $0\leqq x\leqq 1$, $P_n(0)=0$, $P_n(1)=1$, and $\int_0^1 P_n(x)dx$ is a minimum. (II) Find $P_n(x)$ such that $P_n(x)\geqq 0$ in $-1\leqq x\leqq 1$, $P_n(\pm 1)=0$, $P_n(0)=1$, and $\int_{-1}^{1}P_n(x)dx$ is a minimum. (III) Find $P_n(x)$ so that $P_n(-1)=-1$, $P_n(1)=1$, $|P_n(x)|\leqq 1$ and $\int_{-1}^{1}P_n(x)\operatorname{sgn} x dx$ is a maximum. For problems (I) and (II) the minimal polynomials and the minima of the integrals are computed explicitly. For problem (III) only estimates are given. The problems are said to arise in acoustics.

R. P. Boas, Jr. (Providence, R. I.).

Herzog, Fritz, and Hill, J. D. The Bernstein polynomials for discontinuous functions. Amer. J. Math. 68, 109–124 (1946). [MF 15493]

A skeleton is defined as a function $f(r)$ defined only for rational values of r. If the limits $f_L(x)=\lim_{r\to x-0}f(r)$ and $f_R(x)=\lim_{r\to x+0}f(r)$ exist for $0<x\leqq 1$ and $0\leqq x<1$, respectively, then $f(r)$ is said to belong to the class $\mathfrak{S}$ and the function $f_N(x)=\frac{1}{2}\{f_L(x)+f_R(x)\}$, $0<x<1$, $f_N(0)=f(0)$, $f_N(1)=f(1)$ is known as the normalized extension of $f(r)$. The authors prove that for any f of $\mathfrak{S}$ the Bernstein polynomials $B_n(f,x)=\sum_{k=0}^{n}f(k/n)T_{n,k}(x)$, where

$$T_{n,k}(x)=\binom{n}{k}x^k(1-x)^{n-k},$$

converge in [0, 1] to the normalized extension of $f(r)$. They also give inequalities for the upper and lower limits of $B_n(f,x)$ for any bounded skeleton and some sufficient conditions for the convergence of the Bernstein polynomials of any given skeleton $f(r)$.

A. C. Offord.

Calugareanu, Georges. Sur les polynomes de Tchebichef d'un ensemble plan borné et fermé. Bull. Sci. Math. (2) 69, 75–81 (1945). [MF 15415]

The author gives an algorithm for the construction of the Chebyshev polynomials $T_n(z)$ associated with a bounded plane set E. Let $E(r)$ be the plane set formed of all the points of the plane whose distances from any point of E are less than r. If E and E_1 are any two plane sets, then $r=\delta(E,E_1)$ is said to be the span of these two sets if it is the smallest number such that $E(r)$ contains E_1 and $E_1(r)$ contains E. The author proves that, if $\{E_\nu\}$ is a sequence of sets such that $\delta(E,E_\nu)\to 0$ as $\nu\to\infty$, then the Chebyshev polynomials $T_n(z;E_\nu)$ associated with E_ν converge uniformly to $T_n(z)$. The sets E_ν can be finite sets and the final step is to construct the Chebyshev polynomials associated with finite plane sets.

A. C. Offord.

Rainville, E. D. Symbolic relations among classical polynomials. Amer. Math. Monthly 53, 299–305 (1946). [MF 16720]

Numerous relations involving the polynomials of Legendre, Laguerre and Hermite are obtained in the compact symbolic form that upon expansion calls for replacement of certain exponents by corresponding subscripts. The derivations are made by manipulating known symbolic relations, by use of linear differential operators, and by use of generating functions.

I. M. Sheffer (State College, Pa.).

Sansone, G. Su una immediata limitazione delle derivate dei polinomi di Legendre. Boll. Un. Mat. Ital. (2) 4, 145–147 (1942). [MF 16064]

The author establishes two simple (not best possible) inequalities for the derivatives of Legendre polynomials.

R. P. Boas, Jr. (Providence, R. I.).

Palamà, G. Su alcune relazioni limiti relative a classici polinomi. Boll. Un. Mat. Ital. (2) 4, 99–109 (1942). [MF 16061]

The first part of this paper is devoted to remarks on the formula

$$(1)\quad H_n(x)=2^n n!\lim_{a\to 0}\{a^n L_n^{((1/2a^2)+(m/2a)+k)}[((2x+m)/a)+(1/2a^2)]\}.$$

The author [same Boll. (2) 1, 27–35 (1939)] obtained the particular case $k=-1$, $m=0$; L. Toscano [ibid., 337–339 (1939)] generalized Palamà's result by introducing k (but

keeping $m=0$), and now Palamà himself gives the still more general form (1) and derives some results from it.

In the second part of the paper Hermite polynomials are represented as limiting forms of ultraspherical polynomials.

A. Erdélyi (Edinburgh).

Toscano, Letterio. Legami tra formule limite su polinomi classici. Boll. Un. Mat. Ital. (2) 5, 31–34 (1943). [MF 16083]

Further remarks concerning the formulae established by Palamà [cf. the preceding review], mainly to the effect that the more general formulae can be obtained from the special ones by a second limiting process. *A. Erdélyi.*

Toscano, Letterio. Formule di addizione e moltiplicazione sui polinomi di Laguerre. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. 76, 417–432 (1941). [MF 16263]

The addition theorem

$$L_n^{(\alpha_1+\cdots+\alpha_r+r-1)}(x_1+\cdots+x_r)=\sum_{i_1+\cdots+i_r=n} L_{i_1}^{(\alpha_1)}(x_1)\cdots L_{i_r}^{(\alpha_r)}(x_r)$$

for Laguerre polynomials is known. The author derives another addition theorem

$$L_n^{(\alpha_1+\cdots+\alpha_r-n)}(x_1+\cdots+x_r)=\sum_{i_1+\cdots+i_r=n} L_{i_1}^{(\alpha_1-i_1)}(x_1)\cdots L_{i_r}^{(\alpha_r-i_r)}(x_r)$$

by means of generating functions and then gives alternative proofs by means of differential operators of both addition theorems. If all the x_i are equal, a multiplication theorem is obtained. With the help of these and some known relations, numerous series of Laguerre and Hermite polynomials are summed. *A. Erdélyi* (Edinburgh).

Feldheim, E. Su un sistema di polinomi ortogonali a distribuzione del tipo di Stieltjès. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. 77, 526–536 (1942). [MF 16259]

This is an illustration of a general point of view expressed by N. V. Bugaev in the following words. "To almost every important branch of analysis there is a corresponding similar branch of arithmology." A perfect analogue $\{l_n^{(\alpha)}(x)\}$ of the classical system $\{L_n^{(\alpha)}(x)\}$ of orthogonal Laguerre polynomials is defined and studied for the domain of a discontinuous positive integer variable $x=0, 1, 2, \cdots$. The results can be summarized by saying that all the properties of $L_n^{(\alpha)}(x)$ have their analogues in the system $\{l_n^{(\alpha)}(x)\}$. The polynomials $l_n^{(\alpha)}(x)$ depend on a positive parameter λ and, if $\lambda\to 0$, in the limit $l_n^{(\alpha)}(x/\lambda)$ is transformed into $L_n^{\alpha}(x)$. Another similar result is

$$\lim_{n\to\infty} n^{-x} l_n^{(\alpha)}(x, \lambda/n)=e^{-\lambda}x!L_x^{(\alpha)}(\lambda)/\Gamma(x+\alpha+1).$$

E. Kogbetliantz (New York, N. Y.).

Special Functions

Leemans, J. Sur les fonctions de Bessel. Mathesis 54, 242–261 (1940). [MF 15525]

The author derives a collection of well-known results concerning the Bessel functions and their integrals. In this paper the second solution of Bessel's differential equation $y''+x^{-1}y'+(1-n^2/x^2)y=0$ is defined by

$$N_n(x)=\frac{J_n(x)\cos n\pi-J_{-n}(x)}{\sin n\pi},$$

a definition which implies that n is not an integer. This definition is, however, everywhere applied by the author in the case when n is an integer. *S. C. van Veen* (Delft).

Rutgers, J. G. Sur des séries et des intégrales définies contenantes les fonctions de Bessel. V, VI. Nederl. Akad. Wetensch., Proc. 44, 978–988, 1092–1098 (1941). [MF 15764]

Continuation of earlier papers of the same title [same Proc. 44, 464–474, 636–647, 744–753, 840–851 (1941); these Rev. 3, 116, 117] adding over a hundred additional formulas. *M. C. Gray* (New York, N. Y.).

Bose, B. N. On some integrals involving Bessel functions. Bull. Calcutta Math. Soc. 37, 77–80 (1945). [MF 15682]

"The object of the present note is to obtain certain integrals involving Bessel functions. The method adopted is that of the operational calculus." Some of the results are vitiated by a mistake. The author "notes" that the inverse Laplace transform of $p^{-\frac{1}{2}}\mathbf{L}_0(a\sqrt{p})$ ($\mathbf{L}$ is Struve's function) vanishes identically, and states that the same is easily seen to be true of $p^{-1}\mathbf{L}_{2n+1}(a\sqrt{p})$; in fact neither of these functions has an inverse Laplace transform. *A. Erdélyi.*

Sinha, S. A few integrals involving Bessel and hypergeometric functions. Proc. Benares Math. Soc. (N.S.) 6, 3–9 (1944). [MF 16156]

The integrals discussed by the author involve products of confluent hypergeometric functions and modified Bessel functions. The formulas obtained are too complicated to be reproduced here, especially as there are numerous misprints. The method used consists in substituting a known integral for a Bessel function (or for the product of two Bessel functions) under the integral sign and then inverting the order of integration. *M. C. Gray* (New York, N. Y.).

Agostinelli, Cataldo. Sopra alcuni integrali delle funzioni cilindriche. Boll. Un. Mat. Ital. (2) 4, 25–28 (1942). [MF 16052]

The author has [Ann. Mat. Pura Appl. (4) 17, 255–287 (1938)] proved the relation

$$\int_0^\infty J_n(sx)J_n(sy)sds\int_0^\infty J_0(s\sigma)f(\sigma)\sigma d\sigma=(2\pi)^{-1}\int_0^{2\pi} f((x^2+y^2-2xy\cos\theta)^{\frac{1}{2}})\cos n\theta d\theta.$$

In the present paper this relation is applied to a number of functions f for which the integral

$$\int_0^\infty J_0(s\sigma)f(\sigma)\sigma d\sigma$$

can be evaluated explicitly. In this way a number of integrals are obtained; not all the results are new or presented in the simplest form possible. *A. Erdélyi* (Edinburgh).

Feldheim, E. Alcuni risultati sulle funzioni di Whittaker e del cilindro parabolico. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. 76, 541–555 (1941). [MF 16261]

The author proves

$$\lim_{\omega\to\infty}\frac{\Gamma(\omega^2+k+\nu+1)}{\omega^\nu\Gamma(\omega^2+k+1)}\,{}_1F_1(-\nu;\omega^2+k+1;\omega^2-\omega x)=e^{\frac{1}{4}x^2}D_\nu(x)$$

and by means of this relation derives expansions for D_ν from known expansions for ${}_1F_1$. Next he represents $e^{\frac{1}{4}z^2}D_\mu(z)D_\nu(z)$

as a Laplace integral and evaluates

$$\int_{-\infty}^{\infty} e^{iux} D_n(x-v) D_{n+\mu}(x+v)dx$$

in terms of the Laguerre polynomial $L_n^\mu(u^2+v^2)$. Some further results follow from the "generalized generating function"

$$e^{\frac{1}{2}tz-\frac{1}{4}t^2}D_\nu(z-t)=\sum_0^\infty (t^n/n!)D_{\nu+n}(t).$$

The rest of the paper is devoted to the evaluation of infinite integrals containing parabolic cylinder functions.

A. Erdélyi (Edinburgh).

Pinney, Edmund. Laguerre functions in the mathematical foundations of the electromagnetic theory of the paraboloidal reflector. J. Math. Phys. Mass. Inst. Tech. **25**, 49–79 (1946). [MF 16150]

The wave equation is separable in paraboloidal coordinates and the normal solutions involve what the author calls Laguerre functions; $L_\nu^\mu(z)$ reduces to the generalized Laguerre polynomial if ν is a nonnegative integer. For any ν the Laguerre function can be expressed in terms of Whittaker's function $M_{k,m}$ in the form

$$L_\nu^\mu(z)=\frac{\Gamma(\mu+\nu+1)}{\Gamma(\mu+1)\Gamma(\nu+1)} z^{-\frac{1}{2}\mu-\frac{1}{2}}e^{\frac{1}{2}z}M_{\frac{1}{2}\mu+\nu+\frac{1}{2},\frac{1}{2}\mu}(z),$$

so that many of the results given in this paper are transcriptions of known results on Whittaker functions. Since in the solution of the wave equation only the combination $z^{\frac{1}{2}\mu}e^{-\frac{1}{2}z}L_\nu^\mu(z)$ appears, it is not at all clear to the reviewer why Laguerre functions are introduced at all. A second solution $U_\nu^\mu(z)$ of the Laguerre differential equation is defined in terms of L_ν^μ; this second solution is expressible in terms of Whittaker's function $W_{k,m}$.

A considerable number of useful expansions, integrals, asymptotic representations, etc. are given and also formulae for combinations such as

$$L_{\nu+1}^\mu(z)L_\nu^\mu(t)-L_\nu^\mu(z)L_{\nu+1}^\mu(t), \quad L_{\nu+1}^\mu(z)U_\nu^\mu(t)-L_\nu^\mu(z)U_{\nu+1}^\mu(t).$$

Appended to the paper there is an extensive bibliography (by H. Bateman) on Laguerre polynomials. The related work of H. Buchholz [Z. Angew. Math. Mech. **23**, 47–58, 101–118 (1943); Ann. Physik (5) **42**, 423–460 (1942); these Rev. **5**, 182, 249] on the solution of the wave equation in paraboloidal coordinates is not referred to. *A. Erdélyi*.

Poli, Louis. Sinus d'ordre n et fonction $\nu(x)$. C. R. Acad. Sci. Paris **222**, 580–581 (1946). [MF 16035]

The sine of order n is defined by the series

$$f(x, a, n)=\sum_{k=1}^\infty(-1)^{k-1}x^{kn-a}/\Gamma(kn-a+1),$$

with a related definition for the hyperbolic sine of order n, $h(x, a, n)$. The function $\nu(x)$ is defined by the integral

$$\nu(x, n)=\int_n^\infty \frac{x^s ds}{\Gamma(1+s)}, \qquad \nu(x)=\nu(x, 0).$$

The author shows that definite integrals of h and f with respect to either of the parameters a, n may be evaluated as series of functions $\nu(x, n)$. *M. C. Gray*.

Bateman, H. An extension of Schuster's integral. Proc. Nat. Acad. Sci. U. S. A. **32**, 70–72 (1946). [MF 15898]

Put

$$C(x)=\int_x^\infty \cos t^n dt, \quad S(x)=\int_x^\infty \sin t^n dt, \qquad n>1.$$

The author evaluates $\int_0^\infty C(x)C(ax)dx$ and similar integrals in which one or both of the C's are replaced by S's and derives other integrals. The connection of the results with integrals of products of incomplete gamma functions is pointed out. *A. Erdélyi* (Edinburgh).

Differential Equations

Verstraete, Roland. On the conditions for integrability of Riccati differential equations. Wis- en Natuurk. Tijdschr. **12**, 180–182 (1946). (Dutch. French summary) [MF 16750]

Sispánov, Sergio. On a differential equation of 2d order. Revista Unión Mat. Argentina **9**, 165–170 (1943). (Spanish) [MF 16816]

The equation is $[\log\{(1+y')/y'\}]'=x^{-1}+y^{-1}-(x+y)^{-1}$.

Okamura, Hirosi. Sur l'unicité des solutions d'un système d'équations différentielles ordinaires. Mem. Coll. Sci. Kyoto Imp. Univ. Ser. A. **23**, 225–231 (1941). [MF 15819]

We quote some of the results. (A) If a family of curves Ω is given by equations $\Phi_i(x, y_1, \cdots, y_n)=C_i$, $i=1, \cdots, n$, where the Φ_i's satisfy a Lipschitz condition, then the system of differential equations with directions given by Ω has no other solutions than the curves Ω. (B) Given the system

$$(1) \qquad dy_i/dx=f_i(x, y_1, \cdots, y_n)=f(x, y), \quad i=1, \cdots, n,$$

with the f_i continuous in $G: 0\leqq x\leqq a$, $|y_i|<b$, $i=1, \cdots, n$, and with $f_i(x, 0)=0$ for $0\leqq x\leqq a$, $i=1, \cdots, n$, then, in order that the solution of (1) through (0, 0) be unique, it is necessary and sufficient that there exist a function $\phi(x, y)$ such that $\phi(x, 0)=0$ for $0\leqq x\leqq a$, but $\phi(x, y)\neq 0$ for $|y_1|+\cdots+|y_n|\neq 0$, with ϕ satisfying a Lipschitz condition in the y's, and such that, if (x, y) is in G, then

$$\overline{\lim_{t\to 0}}\, t^{-1}\{\phi[x+t, y_1+tf_1(x, y), \cdots, y_n+tf_n(x, y)]-\phi(x, y)\}\leqq 0.$$

The work is based largely on a function $D(P, Q)$ defined as follows. If P is joined to Q by a broken line consisting alternately of segments having the direction of (1) at the end with minimum x, and segments with x constant, let Δ denote the sum of the lengths of the segments for which x is constant. Then $D(P, Q)$ is the greatest lower bound of all possible values of Δ for the given P, Q. In the preceding theorem, the necessity is proved by taking $\phi(x, y)=D(0, P)$.

Another theorem: given that the f_i are continuous, a necessary and sufficient condition that two given points P and Q lie on a single trajectory of (1) is that $D(P, Q)=0$.

A. B. Brown (Flushing, N. Y.).

Okamura, Hirosi. Condition nécessaire et suffisante remplie par les équations différentielles ordinaires sans points de Peano. Mem. Coll. Sci. Kyoto Imp. Univ. Ser. A. **24**, 21–28 (1942). [MF 15817]

Given the system (1) of the preceding review, the author states a condition necessary and sufficient for local uniqueness of solutions of (1) on the right hand side of a given initial point, with corresponding condition for uniqueness on the left side. For the right hand side, the condition is the existence of a function $\Phi(x, y_1, \cdots, y_n, z_1, \cdots, z_n)$ having continuous first partial derivatives, such that $\Phi(x, y, z)$ is

positive or zero according as $|y_1-z_1|+\cdots+|y_n-z_n|$ is positive or zero, and

$$\frac{\partial\Phi}{\partial x}+\sum_{i=1}^{n}\frac{\partial\Phi}{\partial y_i}\cdot f_i(x, y_1, \cdots, y_n)+\sum_{i=1}^{n}\frac{\partial\Phi}{\partial z_i}\cdot f_i(x, z_1, \cdots, z_n)\leqq 0.$$

(For uniqueness on the left side, reverse the last inequality sign.) The sufficiency of the condition is shown by a short simple argument. In the proof of the necessity, the reviewer found the author's exposition clear up to a certain inequality [(6), p. 25], at which point there appears to be either an error or a serious omission. Examples given at the end of the paper are in accordance with the theorem as stated.

A. B. Brown (Flushing, N. Y.).

Satô, Tokui. Sur l'équation différentielle contenant un paramètre. Jap. J. Math. 17, 299–305 (1941). [MF 14960]

The author studies the equation (1) $xdy/dx=f(x, y, \lambda)$, where f is analytic at $x=y=\lambda=0$;

$$f=cy+(a_{00}(\lambda)+a_{10}(\lambda)x+\cdots)+(a_{01}(\lambda)+a_{11}(\lambda)x+\cdots)y+\cdots;$$

(i) $a_{00}(0)=0$, $a_{01}(0)=0$. When (ii) c is not a nonnegative integer, the following has been established previously: (1) has a unique solution, analytic in x, λ and vanishing for $x=\lambda=0$; if $c\geqq 0$, (1) has a solution $\phi(x, u, \lambda)$ analytic for $x=u=\lambda=0$, where $u=\Gamma x^{c+\alpha(\lambda)}$ (any constant Γ; $\alpha(\lambda)$ a certain analytic function; $\alpha(0)=0$). The author establishes an analogous result when (iii) $c=n$ (a positive integer), proving the following. If (1) satisfies (i), (iii), there is a solution $y=\phi(x, u, \lambda)$, analytic in x, u, λ at $x=u=\lambda=0$, with

$$u=x^{n+\alpha(\lambda)}\{C-(a(\lambda)/\alpha(\lambda))(1-x^{-\alpha(\lambda)})\},\qquad a(\lambda)=-a_{n0}(\lambda).$$

W. J. Trjitzinsky (Urbana, Ill.).

Chazy, Jean. Sur une équation différentielle du premier ordre et du second degré. Ann. Sci. École Norm. Sup. (3) 61, 45–71 (1944). [MF 14645]

The author makes a systematic study of the nature of the solutions of the differential equation $Py'^2+Qy'+R=0$ near a singular point; P, Q, R are assumed to be expansible in power series in x, y about the singular point and to vanish at that point. Examples are given of the various cases which can arise and applications are made to the structure of the lines of curvature on a surface in the neighborhood of an umbilical point. The results obtained supplement earlier work of Cayley, Picard and others. [See, for example, Picard, Traité d'Analyse, vol. 3, 3d ed., Gauthier-Villars, Paris, 1928, pp. 223–234.] *W. Kaplan.*

Choquet, Gustave. Caractérisation topologique des équations différentielles $y'=f(x, y)$ admettant un groupe transitif de transformations. C. R. Acad. Sci. Paris 222, 718–719 (1946). [MF 16175]

The author states the following result. Let F be the family of solutions of the differential equation $y'=f(x, y)$, where f is continuous and bounded for all x, y. Let G be the group of all orientation-preserving homeomorphisms T of the plane on itself, such that, for every c in F, $T(c)$ and $T^{-1}(c)$ are in F. If G is transitive, then F is homeomorphic to one of the families of curves $y=\varphi(x)$ defined by (a) $\varphi'(x)=0$; (b) $0\leqq\varphi'(x)\leqq 1$; (c) $0\leqq\varphi'(x)\leqq e^x$; (d) $0\leqq\varphi'(x)\leqq 1/(1+x^2)$. A brief indication of the proof is given. *W. Kaplan.*

Lahaye, Edmond. Les développements des intégrales des équations $dy/dx=P(x, y)/Q(x, y)$ dans le domaine des valeurs qui annulent simultanément P et Q. Acad. Roy. Belgique. Cl. Sci. Mém. Coll. in 8°. (2) 20, no. 5, 123 pp. (1945). [MF 16505]

This paper gives a very elaborate and systematic discussion of those solutions of the equation of the title which pass through a point, say the point (0, 0), at which both of the holomorphic functions P and Q are zero. Many more or less particular cases have been studied previously by Briot and Bouquet, Bendixson, Dulac and others. The present author attacks the problem in what he considers to be its full generality. The greater part of the paper is devoted to a study of the effect upon the form of the equation of the transformation of variables $x_1=x^{1/a}$, $y=x^{b/a}(\rho+y_1)$. Here a and b are integers and ρ is a constant. It is found that the equation can always be reduced, by a finite sequence of such transformations, to some one of seven normal forms. These normal forms are considered to be irreducible.

In the second part of the paper the analytical forms of the solutions of the several normal equations are determined. This part of the discussion necessitates the examination of various subcases. The results of the work consist essentially of a large body of formulas and do not lend themselves to any concise restatement. *L. A. MacColl.*

Haag, Jules. Sur la stabilité des solutions de certains systèmes d'équations différentielles. C. R. Acad. Sci. Paris 222, 623–624 (1946). [MF 16039]

Consider the system of equations

$$udy_j/du=S_j(y_j+y_{j-1})+g_j(u, y_k),\quad j=1, \cdots, m.$$

The S's are constants. The functions g_j are continuous in the domain $0\leqq u\leqq u_0$, $|y_j|\leqq b$, and they have continuous first partial derivatives with respect to the y's in that domain. The functions and their partial derivatives all vanish at the point $u=y_1=\cdots=y_m=0$. Let s_j denote the real part of S_j.

In this note the following results are stated without proof. (1) If all of the s_j's are positive, all solutions of the differential equations are stable, in the sense that all of the y's approach zero with u. (2) If some of the s_j's are negative, there exist solutions for which some $|y_j|$ remains greater than b as u tends toward zero. [The meaning of this is not entirely clear.] (3) If certain of the s_j's are zero, it can happen, with suitable forms of the functions g_j, that the $|y_j|$'s remain less than b but do not necessarily approach zero as u tends toward zero. The discussion is extended to the case of systems of equations of the forms

$$f(x)dy_j/dx=S_j(y_j+y_{j-1})+f_j(x, y_k)$$

and

$$dx_j/dt=r_j(x_j+x_{j-1})+\varphi_j(x_1, \cdots, x_m).$$

In connection with the last system of equations, the discussion results in a known theorem of Liapounoff concerning stability. *L. A. MacColl* (New York, N. Y.).

Haag, Jules. Sur le régime transitoire précédant la synchronisation. C. R. Acad. Sci. Paris 222, 314–316 (1946). [MF 16005]

This note relates to the motion of a nonlinear dynamical system of one degree of freedom subjected to a perturbing force which is a sinusoidal function of time; x denotes the instantaneous phase difference between the oscillatory motion and the perturbing force, and y denotes the instantaneous amplitude of the oscillation. Appropriate simplify-

ing assumptions lead to the approximate equations

$$2T dx/dt = \epsilon + ky^{-1} \cos x, \quad 2T dy/dt = k \sin x - f(y),$$

where T, ϵ, and k are constants, and $f(y)$ is a given function. Then we have the relation

$$y\{k \sin x - f(y)\} dx = \{\epsilon y + k \cos x\} dy.$$

The qualitative properties of the general solution of the last equation are discussed in terms of the locations and natures of the singular points. Various physical implications of the results are indicated. *L. A. MacColl* (New York, N. Y.).

Wintner, Aurel. Asymptotic integration constants in the singularity of Briot-Bouquet. Amer. J. Math. **68**, 293–300 (1946). [MF 16426]

This paper is concerned with the solutions of the real Briot-Bouquet equation $xy' = py + qx + \varphi(x, y)$ in the neighborhood of the point $(0, 0)$ in the case in which that point is a node. In any such case the equation can be reduced to one of three normal forms, in which we have, respectively, $p=1$, $q=0$; $p>1$, $q=0$; and $p=q=1$. The principal results can be stated as follows. On a rectangle

$$(1) \qquad 0<x<a, \quad -b<y<b,$$

let $\varphi(x, y)$ be a real-valued continuous function, subject to the restriction

$$(2) \qquad \int_{+0} x^{-1-p} |\varphi(x, 0)| dx < \infty$$

(where p is some positive number) and, for sufficiently small $x>0$, to the Lipschitz condition

$$|\varphi(x, y_1) - \varphi(x, y_2)| \leqq \mu(x) |y_1 - y_2|,$$

where $\mu(x)$ is defined (and, for instance, continuous) for small $x>0$ and satisfies

$$\int_{+0} x^{-1} \mu(x) dx < \infty.$$

Then, if a and b in (1) are chosen small enough, the behavior of the solutions of

$$(3) \qquad xy' = py + \varphi(x, y)$$

can be described as follows. Every solution path $y=y(x)$ of (3), issuing from an arbitrary point (x_0, y_0) of the rectangle (1), exists on the whole interval $0<x<x_0$ and tends to the point $(0, 0)$ in such a way that there exists a constant $c=c(x_0, y_0)$ satisfying

$$(4) \qquad y(x) = cx^p + o(x^p), \qquad x \to +0.$$

Conversely, if the (real) value of the integration constant c is assigned arbitrarily, there belongs to it a unique solution path $y=y(x)$ satisfying (4).

The above assertions remain true if (2), (3), (4) are replaced by

$$(2') \qquad \int_{+0} x^{-2} |\varphi(x, x \log x)| dx < \infty,$$

$$(3') \qquad xy' = y + x + \varphi(x, y),$$

$$(4') \qquad y(x) = x \log x + cx + o(x), \qquad x \to +0,$$

respectively.

In these theorems the hypotheses are weaker and the conclusions are more precise than in the corresponding theorems given by Perron [Math. Z. **15**, 121–146 (1922); **16**, 273–295 (1923)]. Indeed, it is shown that the present results are, in a certain sense, the best possible.

L. A. MacColl (New York, N. Y.).

Hartman, Philip, and Wintner, Aurel. On the asymptotic behavior of the solutions of a non-linear differential equation. Amer. J. Math. **68**, 301–308 (1946). [MF 16427]

The authors generalize results of Perron for the differential equation $\phi(x)y' = f(x, y)$. In $0<x\leqq a$, $\phi(x)$ is positive and continuous and the integral of its reciprocal diverges. In the rectangle R, $0\leqq x\leqq a$, $-b\leqq y\leqq b$, let $f(x, y)$ be continuous. Let $f(0, 0)=0$. If, for each fixed x, $f(x, y)$ is an increasing function of y in R, then R contains a rectangle R', $0<x<a'$, $-b\leqq y\leqq b$, in which every solution $y=y(x)$ passing through a point (x_0, y_0) of R' can be continued for all positive $x<x_0$, and $y(x)\to 0$ as $x\to +0$ for all continuations. If, on the other hand, $f(x, y)$ is a decreasing function of y for fixed x, then there is one and only one solution of the differential equation for which $y(x)\to 0$ as $x\to +0$. Other results are given. *N. Levinson* (Cambridge, Mass.).

Jounin, Henri. Sur le calcul des fréquences propres des systèmes non linéaires. C. R. Acad. Sci. Paris **222**, 1203–1205 (1946). [MF 16751]

The perturbation method is applied to the equation $md^2x/dt^2 = -m\omega^2 x - m\epsilon g'(x)$, giving solutions up to the first order in ϵ. It is verified that these solutions are isochronous when a sufficient condition given by Chazy [same C. R. **213**, 93–98 (1941); these Rev. **5**, 78] is satisfied.

P. Franklin (Cambridge, Mass.).

Mandelstam, L. I. Systems with periodical coefficients with many degrees of freedom and small non-linearity. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. **15**, 605–612 (1945). (Russian. English summary) [MF 15634]

A nonlinear system of differential equations is considered which differs from a linear system only by terms involving a small parameter. The independent variable appears only in terms which are periodic with a common period. A periodic solution is obtained by the perturbation method.

N. Levinson (Cambridge, Mass.).

Rosenblatt, Alfred. On the phenomenon of subresonance. Case of the generalized van der Pol equation with forced vibrations. Bol. Fac. Ingen. Montevideo **3** (Año 10), 116–126 (1945) = Facultad de Ingeniería Montevideo. Publ. Inst. Mat. Estadistica **1**, 99–109 (1946). (Spanish) [MF 16167]

Using the method of a small parameter, the author finds a sufficient condition for the differential equation

$$x'' + w_n^2 x + \{(A + Cx^2)x' + A_2 x^2 + A_3 x^3\} = e \cos wt + f \sin wt,$$

with $w=2w_n$, to have a periodic solution of period $2\pi/w_n = 4\pi/w$. This condition consists of the two inequalities

$$4ACw_n^2 + M < 0,$$
$$A^2 w_n^2 (w_n^2 - w^2) M < Mk^2 A_2^2 + 2k^2 A^2 C^2 w_n^4,$$

where $M = 2k^2(C^2 w_n^2 + 9A_3^2)/(w_n^2 - w^2)^2$ and $k^2 = e^2 + f^2$, and the requirement that a certain determinant should not vanish. *N. Levinson* (Cambridge, Mass.).

Matos Peixoto, Mauricio. On the solutions of the equation $yy'' = \phi(y')$ which pass through two points of the half-plane $y>0$. Revista Unión Mat. Argentina **11**, 84–91 (1946). (Spanish) [MF 15672]

The following theorem is established. Let a curve c be given, of the form $y=f(x)$, $x_1\leqq x\leqq x_2$, satisfying the conditions $f(x)>0$, $f(x)$ is of class C', $f(x_1)=f(x_2)=0$, $f'(x_1+0) = -f'(x_2-0) = \infty$, $f'(x)$ is monotone strictly decreasing (or increasing); let F denote the family of curves obtained

from c by an arbitrary similarity transformation with respect to the origin followed by a translation parallel to the x-axis. Then there is a unique member of F through each pair of points A, B having unequal abscissas in the half-plane $y>0$. Applications are made to solutions of the differential equation $yy''=\varphi(y')$. *W. Kaplan.*

Cimino, Massimo. Sul comportamento asintotico degli integrali di una equazione delle dinamica. Boll. Un. Mat. Ital. (2) 5, 78–87 (1943). [MF 16090]

In the second order linear equation $y''+p(x)y'+q(x)y=0$, suppose

$$-\infty \leqq \int_a^\infty p(x)dx < +\infty.$$

Let $y_1(x)$ and $y_2(x)$ be independent solutions which are bounded as $x\to+\infty$. Then the perturbed equation

$$y''+p(x)y'+\{q(x)+Q(x)\}y=0$$

will also have bounded solutions as $x\to+\infty$ if

$$\int_a^\infty |Q(x)y_i(x)|dx<\infty, \qquad i=1, 2.$$

N. Levinson (Cambridge, Mass.).

Nagumo, Mitio. Über das Randwertproblem der nicht linearen gewöhnlichen Differentialgleichungen zweiter Ordnung. Proc. Phys.-Math. Soc. Japan (3) 24, 845–851 (1942). [MF 15046]

The author gives sufficient conditions for the existence of at least one solution of the second order differential equation $d^2y/dx^2=f(x, y, dy/dx)$ which satisfies the boundary conditions $y(a)=b$, $y(a_1)=b_1$. Due to their complication, the conditions will not be detailed here; in general, however, they are of the "comparison type." *W. T. Reid.*

Nagumo, Mitio. Eine Art der Randwertaufgabe von Systemen gewöhnlicher Differentialgleichungen. Proc. Phys.-Math. Soc. Japan (3) 25, 221–226 (1943). [MF 15056]

This paper is concerned with existence theorems of Perron type for a so-called "Hukuhara problem" consisting of a system of first order differential equations and initial conditions

$$(*) \quad dy_\mu/dx=F_\mu(x, y_1, \cdots, y_m), \quad y_\mu(a_\mu)=b_\mu, \quad \mu=1, \cdots, m,$$

where $a_1, \cdots, a_m$ are points of the interval range of the real independent variable x. The author gives two general existence theorems which are concerned with the respective cases of real- and complex-valued dependent variables $y_1, \cdots, y_m$. These theorems extend previous results of Hukuhara [J. Fac. Sci. Hokkaido Imp. Univ. Ser. I. 2, 13–88 (1934)] for a system of the form (*).

W. T. Reid (Evanston, Ill.).

Nagumo, Mitio. Eine Art der Randwertaufgabe von Systemen gewöhnlicher Differentialgleichungen. II. Proc. Phys.-Math. Soc. Japan (3) 25, 384–390 (1943). [MF 15059]

The existence theorems of the paper reviewed above for a differential system of the form (*) are generalized here, especially in regard to the type of region involved; in addition, the author establishes uniqueness theorems for such a system. *W. T. Reid* (Evanston, Ill.).

Nagumo, Mitio. Eine Art der Randwertaufgabe von Systemen gewöhnlicher Differentialgleichungen. III. Proc. Phys.-Math. Soc. Japan (3) 25, 615–616 (1943). [MF 15072]

[Cf. the preceding review.] This is an abstract of existence theorems for a "Hukuhara problem" in which the independent variable is complex. *W. T. Reid.*

Sansone, G. Su un criterio sufficiente di esistenza e di unicità per una classe di problemi ai limiti relativi alle equazioni differenziali lineari omogenee di quarto ordine. Boll. Un. Mat. Ital. (2) 5, 72–78 (1943). [MF 16089]

It is shown that a differential equation $y^{(iv)}+4p_1(x)y'''+6p_2(x)y''+4p_3(x)y'+p_4(x)y=0$ may be written in the form

$$(*) \quad [\theta_2(x)y'']''-[\theta_1(x)y']'-\omega(x)y'+\theta_0(x)y=0, \quad a\leqq x\leqq b,$$

whenever $p_1(x), \cdots, p_4(x)$ satisfy suitable differentiability conditions on $a\leqq x\leqq b$. With the aid of a device used by Picard [Traité d'Analyse, vol. 3, 3d ed., Gauthier-Villars, Paris, 1928, pp. 97–99] in a similar problem for second order equations, the author determines a quantity Q dependent upon $\min \theta_2(x)$, $\max|\theta_1(x)|$, $\max|\theta_0(x)|$ and $\max|\omega(x)|$ on $a\leqq x\leqq b$ such that if $0<b-a<Q$ there is a unique solution of (*) satisfying boundary conditions $y(a)=A$, $y'(a)=A'$, $y(b)=B$, $y'(b)=B'$. The author points out that the method of proof employed has been used by Boulanger [Bull. Soc. Roy. Sci. Liége 9, 110–116 (1940); these Rev. 7, 159] for equations of the third order. He does not refer to a later paper of Boulanger [ibid. 11, 220–233 (1942); these Rev. 7, 159] on fourth order equations, however, and in view of the inaccessibility of this paper to the reviewer it is impossible to state the relation of Boulanger's results to those of the present paper. *W. T. Reid.*

Cole, Randal H. Reduction of an n-th order linear differential equation and m-point boundary conditions to an equivalent matrix system. Amer. J. Math. 68, 179–184 (1946). [MF 15500]

The paper is concerned with a differential system

$$(1) \qquad \frac{d^nu}{dx^n}+P_1(x, \lambda)\frac{d^{n-1}u}{dx^{n-1}}+\cdots+P_n(x, \lambda)u=0,$$

$$\sum_{\mu=1}^m V_i^{(\mu)}(u, \lambda)=0, \qquad i=1, 2, \cdots, n,$$

in which

$$P_k(x, \lambda)=\sum_{l=0}^k P_{k,l}(x)\lambda^l,$$

and in which each $V_i^{(\mu)}(u, \lambda)$ is a linear form in the values of u and its first $n-1$ derivatives at the μth point of a set $a_1, a_2, \cdots, a_m$ on the fundamental interval (a, b). Specifically, the matter at issue is the reduction of the system to one of the first order expressible in the form

$$(2) \qquad (d/dx)y=\{\lambda R+Q\}y, \quad \sum_{\mu=1}^m W^{(\mu)}y(a_\mu)=0,$$

$y(x)$ being a vector and R, Q, and $W^{(\mu)}$ being square matrices of order n. The matrix $R(x)$ is of the form $(\delta_{i,j}r_i(x))$, the $r_i(x)$ being the roots of the equation

$$r^n+P_{1,1}(x)r^{n-1}+\cdots+P_{n,n}(x)=0.$$

These roots are assumed to be distinct on (a, b). The reduction makes facts known for systems of the type (2) available for systems of the type (1), and vice versa.

R. E. Langer (Madison, Wis.).

Reid, William T. A matrix differential equation of Riccati type. Amer. J. Math. 68, 237–246 (1946). [MF 16420]

The author considers the equation

$$(1)\qquad W'+WA(x)+D(x)W+WB(x)W=C(x),$$

where $A(x)$, $B(x)$, $C(x)$ and $D(x)$ are given $n\times n$ square matrices of continuous functions. Generalizations of well-known theorems for ordinary Riccati equations are derived. It is shown that a matrix solution $Y(x)$, $Z(x)$ of the system

$$Y'=A(x)Y+B(x)Z,\quad Z'=C(x)Y-D(x)Z$$

exists with $Y(x)$ nonsingular if and only if $W(x)=Z(x)Y^{-1}(x)$ satisfies (1). The second variation for a nonsingular nonparametric fixed end point problem in the calculus of variations is of the form

$$I_2[\eta]=\int_{x_1}^{x_2}[\tilde{\eta}'R(x)\eta'+2\tilde{\eta}'Q(x)\eta+\tilde{\eta}P(x)\eta]dx,$$

where η is an $n\times 1$ matrix and $\tilde{\eta}$ is its transpose. The matrix $R(x)$ is nonsingular. The accessory differential equations for the calculus of variations problem are of the form

$$\eta'=A(x)\eta+B(x)\zeta,\quad \zeta'=C(x)\eta-A(x)\zeta,$$

where $A=-R^{-1}Q$, $B=R^{-1}$, $C=P-\tilde{Q}R^{-1}Q$, and the corresponding Riccati type equation is

$$W'+[\tilde{Q}(x)-W]R^{-1}(x)[Q(x)-W]-P(x)=0.$$

J. E. Wilkins, Jr. (Buffalo, N. Y.).

Bochner, S. Linear partial differential equations, with constant coefficients. Ann. of Math. (2) 47, 202–212 (1946). [MF 16330]

The paper deals with generalized solutions of a linear partial differential equation (1) $\Lambda f=0$ of order N_0 with constant coefficients. Two definitions of "weak" solutions f are shown to be equivalent. The first definition of a weak solution f requires for every neighbourhood U the existence of a sequence of "strict" solutions f_k of (1) of class $N\geqq N_0$, such that, for every bounded measurable function $\psi(x_1,\cdots,x_n)$,

$$\lim_{k\to\infty}\int_U f_k(x)\psi(x)dx_1\cdots dx_n=\int_U f(x)\psi(x)dx_1\cdots dx_n.$$

The second definition of a weak solution f requires that, for every function φ of class C^N vanishing outside an open set D',

$$\int_{D'} f(\bar{\Lambda}\varphi)dx_1\cdots dx_n=0,$$

where $\bar{\Lambda}$ is the adjoint operator to Λ. The notion of weak solution is extended to systems of differential equations with constant coefficients, yielding among other applications a generalization of the theorem of Morera.

The author next discusses "removable singularities." A weak solution $f=(f_1,\cdots,f_r)$ of a system of equations may be defined in a bounded open set D with the exception of a set A. Sufficient conditions are given under which the function obtained by continuing f as identically equal to zero in A represents a weak solution of the system. Generalizations to linear equations with nonconstant coefficients are indicated, using the second definition of weak solution only. In this connection the reviewer wishes to point out a related result by K. O. Friedrichs [Trans. Amer. Math. Soc. 55, 132–151 (1944); these Rev. 5, 188], according to which there exists for every f, to which a system of first order operators can be applied weakly, a sequence f_k, to which the system applies strictly, and which converges towards f (with respect to a certain norm). *F. John.*

Staniukovich, K. P. On automodel solutions of equations of hydrodynamics possessing central symmetry. C. R. (Doklady) Acad. Sci. URSS (N.S.) 48, 310–312 (1945). [MF 16657]

The author reduces the system of equations

$$\partial u/\partial t+u\partial u/\partial r+\rho^{-1}\partial p/\partial r=0,$$
$$\partial\log\rho/\partial t+u\partial\log\rho/\partial r+\partial u/\partial r+Nu/r=0,$$
$$\partial S/\partial t+u\partial S/\partial r=0,\qquad p=\rho^\gamma S,$$

to one ordinary first-order equation by the substitutions

$$u=(t+\tau)^{a_1-1}\xi(z),\quad \rho^{\gamma-1}=(t+\tau)^{a_2}\eta(z),\quad S=(t+\tau)^{a_3}\sigma(z),$$
$$x=\xi/z,\quad y=\eta\sigma/z^2,\quad z=r(t+\tau)^{-a_1},$$

obtaining

$$\frac{(a_1-x)dy-(\gamma-1)ydx}{y\{N(\gamma-1)+(\gamma+1)x-2\}}=\frac{\{(a_1-x)^2-\gamma y\}dx}{y\{2(a_1-1)+a_2/(\gamma-1)+\gamma(N-1)x\}-x(1-x)(a_1-x)},$$

where $\log S$ is proportional to the entropy, $N+1$ is the dimensionality of the flow and $a_3=2(a_1-1)-a_2$. A similar result is obtained with exponential substitutions. The author states that a similar method may be applied to any system of homogeneous equations. *C. C. Torrance.*

Reeb, Georges. Sur les points singuliers d'une forme de Pfaff complètement intégrable ou d'une fonction numérique. C. R. Acad. Sci. Paris 222, 847–849 (1946). [MF 16280]

The author considers a certain type of singular point of a Pfaffian form which he calls a center (a singular point is one where all coefficients of the form vanish). Theorem: if ω is a completely integrable form with only centers as singularities in a manifold V_n, then V_n is a sphere, all integral manifolds are spheres, and there are exactly two centers. Let f be a real valued twice continuously differentiable function on a compact V_n with isolated singularities (points where the first derivatives vanish); assume that different singular points belong to different values of f. Theorems: the hyperspace K_1, whose points are the components of the level surfaces of f, is a finite, 1-dimensional, simplicial complex; its vertices correspond to the singular values of f; the fundamental group of K_1 is a factor group of that of V_n; the inverse image of the interior of a 1-simplex is homeomorphic to the product of an open segment with the inverse image of a point; the order of a vertex of K_1 corresponding to a singular point of index 0 or n is 1, and conversely; similar results for other indices and relations between the numbers of singularities with different indices are given.

H. Samelson (Ann Arbor, Mich.).

Sintzov, D. Untersuchungen über die Theorie der Pfaff'schen Mannigfaltigkeiten (spezielle Mannigfaltigkeiten 1-er u. 2-er Art). Nauk.-Doslid. Inst. Mat. Meh. Harkiv. Univ. Geometričnii Zbirnik 2, 63–83 (1940). (Russian. German summary) [MF 16945]

In Anlehnung an frühere Untersuchungen über Pfaff'sche Mannigfaltigkeiten betrachtet der Verfasser "spezielle" Pfaff'schen Mannigfaltigkeiten (d.h. den Fall der Integrabilität und den Fall, wenn sämtliche Ebenen des Systems eine Fläche berühren). Es werden hinzugefügt: (1) Übergang zu homogenen Koordinaten, (2) Anwendung der Legendre'schen Transformation, (3) besonderer Fall der geodätischen Linien der Gleichung. *Author's summary.*

Sintzov, D. La théorie géométrique de l'intégration des équations différentielles aux dérivées partielles du 1-er ordre à deux variables indépendantes (théorie des caractéristiques). Nauk.-Doslid. Inst. Mat. Meh. Harkiv. Univ. Geometričniĭ Zbirnik 2, 33–61 (1940). (Russian. French summary) [MF 16944]

L'introduction systématique du point de vue géométrique et l'utilisation du principe de dualité permet de préciser l'exposition, notamment en ce qui concerne l'intégrale-planoide du P. du Bois Reymond. Après avoir rappelé la forme géométrique que donne Joukovsky aux problème d'intégration des systèmes normaux, des équations partielles linéaires et non-linéaires l'auteur donne quelques remarques sur les équations de Pfaff, puis passe aux caractéristiques, à la méthode de Cauchy, aux cas singuliers, aux variétés caractéristiques, aux relations entre les équations aux dérivées partielles du premier ordre et les équations de Monge. Au dernier paragraphe les considérations précédentes sont étendues au cas de n variables indépendantes.

Author's summary.

Saltykow, Nicolas. Généralisation des théorèmes de Jacobi et de Poisson. C. R. Acad. Sci. Paris 222, 127–128 (1946). [MF 15986]

The system of linear partial differential equations in n variables, (1) $X_\alpha f=0$ and (2) $X_i f=0$, $\alpha=1, \cdots, m$; $i=m+1, \cdots, m+m'<n$, is said to be semi-normal if X_α is in involution with X_β and X_i while the subsystem (2) is closed. The generalization of the theorem of Jacobi is: if the two sets (1) and (2) are semi-normal, the operator of one applied to an integral of the other determines an integral of the latter. The proof is obvious. The author applies this to a normal system $F_i(x_1, \cdots, x_n; p_1, \cdots, p_n)=0$, $i=1, \cdots, m$; $p_k=\partial z/\partial x_k$. If the linear equations for the characteristics have m' distinct integrals $f_j, j=1, \cdots, m'\leqq n-m$, forming a function group which is assumed to be free of singular functions, the following theorem is proved. The parenthesis of Poisson formed out of F_i and a differential invariant of the above function group is again a differential invariant of the function group.

M. S. Knebelman.

Potier, Robert. Sur les systèmes d'équations aux dérivées partielles linéaires et du premier ordre, à quatre variables, invariants dans toute transformation de Lorentz. C. R. Acad. Sci. Paris 222, 638–640 (1946). [MF 16045]

The author presents three sets of equations, using a notation of É. Cartan [Leçons sur la théorie des spineurs, Actual. Sci. Ind., no. 643, Hermann, Paris, 1938]. He then says that any system of partial differential equations of the type described in the title can be completely decomposed into subsystems in such a way that each subsystem is like one of the three sets of equations presented in the first part of the paper.

A. Schwartz (State College, Pa.).

af Hällström, Gunnar. Eine hinreichende Bedingung der Irregularität eines Randpunktes in Bezug auf die Greensche Funktion in der Ebene. Acta Acad. Aboensis 14, no. 8, 10 pp. (1943). [MF 15101]

The paper contains the proof of the following criterion for an irregular point: if the boundary is contained in a sequence of circles with radii Δ_n and centers at distances r_n from O, where $\Delta_n\leqq r_n$ and $r_{n+1}\leqq r_n$, then O is an irregular point of the boundary if $\sum \log r_n/\log \Delta_n$ converges.

František Wolf (Berkeley, Calif.).

af Hällström, Gunnar. Reguläre und irreguläre Randpunkte der Greenschen Funktion in der Ebene. Acta Soc. Sci. Fennicae. Nova Ser. A. 3, no. 5, 22 pp. (1944). [MF 14657]

The author gives a detailed proof in two dimensions of Bouligand's theorem [Ann. Soc. Polonaise Math. 4, 59–112 (1926)] which, with the usual definition of regularity and irregularity for a boundary point (i) by means of the behavior of the Green's function, (ii) by means of solutions of the Dirichlet problem with continuous boundary values, asserts the equivalence of the two definitions. The second part of the paper is concerned with Wiener's criterion and other criteria for the regularity of a point. Many new ones are given and their necessity and sufficiency are discussed. A typical criterion is the following: if the boundary contains circular arcs of length α_n with radius ρ_n and center at O then O is regular if $\sum\alpha_n\{\rho_n-\rho_{n-1}\}/\rho_n^2$ diverges.

František Wolf (Berkeley, Calif.).

Brelot, Marcel. Sur le problème de Dirichlet ramifié et la représentation conforme. C. R. Acad. Sci. Paris 222, 851–852 (1946). [MF 16282]

In an earlier note the author considered the Dirichlet problem for a domain in a compact space of $\tau\geqq 2$ dimensions, and discontinuous boundary values depending upon the manner of approach [same C. R. 221, 654–656 (1945); these Rev. 7, 204]. The author recalls his extension of the definition of interior and exterior harmonic measure, now gives an alternative generalization and states without proof results concerning the two concepts. He also comments on the relation of the new form of the Dirichlet problem to the theory of conformal mapping.

F. W. Perkins.

Deny, Jacques. Sur l'espace des distributions d'énergie finie et un théorème de H. Cartan. C. R. Acad. Sci. Paris 222, 1374–1376 (1946).

The author considers the potential due to magnetic distributions in space of $n\geqq 3$ dimensions. He defines the energy of such distributions and states a theorem relating to the completeness of a related abstract space. He calls attention to the close connection between this note and the paper by H. Cartan reviewed below.

F. W. Perkins.

Cartan, Henri. Théorie du potentiel newtonien: énergie, capacité, suites de potentiels. Bull. Soc. Math. France 73, 74–106 (1945). [MF 14177]

This paper contains the development of results announced in an earlier note [C. R. Acad. Sci. Paris 214, 944–946 (1942); these Rev. 5, 146]. See also the following review.

J. W. Green (Los Angeles, Calif.).

Cartan, Henri. Sur les fondements de la théorie du potentiel. Bull. Soc. Math. France 69, 71–96 (1941). [MF 13240]

In a later paper, which we shall call H [Bull. Sci. Math. (2) 66, 126–132, 136–144 (1942); these Rev. 6, 86], the author defines the potential of a mass-distribution μ for a base-function f in a homogeneous space defined by a locally compact group and a compact subgroup and formulates four properties (1)–(4) of the potential. The review of H requires the following corrections: (a) the author assumes the base-function f to be such that the measure $f(\dot{x})d\dot{x}$ is symmetric; (b) H does not purport to prove (1)–(4); what it states and proves is that, whenever f is such that one of the properties (1), (2), (3) holds for every μ, the same is true of the other two, and that (4) then holds for the base-

function $f*f$ and every μ; (c) H proves that (1) holds whenever f belongs to a one-parameter family f_α of base-functions ($0<\alpha\leqq 1$), such that: (A) $f_\alpha(\dot{x})d\dot{x}$ is a symmetric measure for every α; (B) for every continuous φ, vanishing outside a compact set, $\int\varphi(\dot{x})f_\alpha(\dot{x})d\dot{x}$ tends to $\varphi(0)$ for $\alpha\to 0$; (C) $f_{\alpha+\beta}=f_\alpha * f_\beta$; this implies, because of (b), that (2), (3), (4) hold under the same assumptions. For some of the proofs, H refers to the paper F under review; otherwise F is superseded by H, except that it gives applications of the above properties to the "balayage" method (the "balayage" of an arbitrary positive mass-distribution, to a compact set, being obtained by minimizing a distance in a suitable Hilbert space), to the definition of capacity, and (under additional assumptions) to the maximum principle of potential-theory. [For later results by the author, cf. the preceding review.]

A. Weil (São Paulo).

Kametani, Syunzi. Positive definite integral quadratic forms and generalized potentials. Proc. Imp. Acad. Tokyo 20, 7–14 (1944). [MF 14866]

L'auteur étend la théorie du potentiel de Frostman [Comm. Sém. Math. Univ. Lund [Medd. Lunds Univ. Mat. Sem.] 3 (1935)] de l'espace euclidien à l'espace topologique métrique de base dénombrable. Il part d'une fonction de deux points p, q (généralisant $1/r$), qui est symétrique, non négative, continue, infinie seulement si $p=q$ et telle que l'intégrale d'énergie d'une distribution de masses de signes quelconques soit non négative et nulle seulement avec la distribution. Il opère dans l'espace des distributions d'énergie finie, muni d'une norme égale à la racine carrée de l'énergie, et se sert de la convergence faible des distributions de masses et d'extractions de suites de distributions bornées. Il peut alors pour un ensemble borélien reprendre la notion de capacité (dite maintenant intérieure), la théorie du balayage pour un compact en minorant l'intégrale de Gauss et l'étude de la distribution obtenue.

[Ces considérations sont connues en France, d'ailleurs pour des espaces plus généraux, depuis 1940; mais on hésitait à mettre dans les hypothèses la propriété fondamentale de l'énergie qui représentait dans la théorie de Frostman et de de la Vallée Poussin la partie la plus difficile. Aussi a t-on attendu des progrès pour que paraissent sur ce sujet des articles de H. Cartan [voir les deux analyses précédentes et Bull. Sci. Math. (2) 66, 126–132, 136–144; ces Rev. 6, 86]. On y trouve des hypothèses générales comprenant celles du cas classique et qui permettent de démontrer ces propriétés-clefs de l'énergie; la théorie y est par ailleurs renouvelée et libérée des extractions de suite grâce à l'emploi de la convergence forte et de la propriété nouvelle de complétion de l'espace des distributions non négatives, d'énergie finie sur un compact.]

M. Brelot (Grenoble).

Bergman, Stefan. A class of harmonic functions in three variables and their properties. Trans. Amer. Math. Soc. 59, 216–247 (1946). [MF 15651]

L'auteur étudie la fonction harmonique du point X (x_1, x_2, x_3):

$$h(x)=\int_L E(X,\zeta)f(\zeta)d\zeta,$$

où f est analytique, L une certaine courbe du plan complexe ζ et

$$E(X,\zeta)=\left\{\sum_{k=1}^{k=3}(x_k-u_k(\zeta))^2\right\}^{-\frac{1}{2}},$$

où les $u_k(\zeta)$ sont des fonctions rationnelles de ζ. On précise les domaines d'harmonicité, le rôle de L qu'on peut faire varier et on étudie spécialement le cas où f est rationnelle, car h devient alors une intégrale hyperelliptique qu'on décompose.

M. Brelot (Grenoble).

Morse, Marston. The topology of pseudo-harmonic functions. Duke Math. J. 13, 21–42 (1946). [MF 15871]

This paper considers the critical points of a harmonic function of two independent variables defined over a limited open region G bounded by ν Jordan curves. More generally, to emphasize the topologic aspects of the theory, the results are applicable to pseudo-harmonic functions. These are defined by the author as functions which are everywhere locally harmonic after a suitable local topologic mapping is performed. Moreover, it is advantageous in these questions to consider pseudo-harmonic functions since the purely topologic aspects are more clearly envisioned and arbitrary topologic transformations may be performed without leaving the class of pseudo-harmonic functions. The function $U(x, y)$ under consideration is pseudo-harmonic over G with a finite number M of logarithmic poles and is assumed to be continuous on the closure of G. Furthermore, as a function defined on the ν boundary curves of G, the boundary values of $U(x, y)$ are assumed to have at most a finite number of maxima and minima. The author shows that then

$$(*)\qquad M-n+m=2-\nu,$$

where m is the number of points (on the boundary) which afford relative minima to $U(x, y)$, and n is the number of saddle points counted with their multiplicities (with appropriate definition of saddle points on the boundary). The essential new point in the proof of (*) is the analysis of the behavior of $U(x, y)$ in the neighborhood of boundary points and, in particular, the proof that n is finite so that no saddle points cluster at the boundary. With this boundary behavior clarified, the relation (*) is essentially the Morse equality $M_2-M_1+M_0=R_2-R_1+R_0$ (related to the Birkhoff minimax principle).

M. Shiffman (New York, N. Y.).

Hardy, G. H., and Rogosinski, W. W. Theorems concerning functions subharmonic in a strip. Proc. Roy. Soc. London. Ser. A. 185, 1–14 (1946). [MF 15182]

The main theorem of the paper is the following. If $f(x, y)$ is subharmonic and bounded above in the half-strip $\alpha<x<\beta$, $y>\gamma$, then either $\bar{\phi}(x)=\lim\sup_{y\to\infty} f(x,y)\equiv-\infty$ or $\bar{\phi}(x)$ is continuous and convex for $\alpha<x<\beta$. The proof is straightforward. The Phragmén-Lindelöf theorem is proved for the half-strip and diverse corollaries are deduced. In the second part it is proved that, if $f(x, y)$ is subharmonic for $\alpha<x<\beta$, then $I(x,y)=\int_{-y}^{y} f(x,y)dy$ is subharmonic in $\alpha<x<\beta$, $y>0$. Several other similar results for integrals of subharmonic functions are deduced.

František Wolf.

Walker, A. G. Note on pseudo-harmonic functions in space of constant curvature. J. London Math. Soc. 20, 32–39 (1945). [MF 15932]

Let Δ^* be the second Beltrami differential operator for a Riemannian 3-space of constant curvature K, and λ a given constant. The author calls a function u pseudo-harmonic in a domain D if it has continuous second derivatives in D and satisfies there the differential equation $\Delta^*u-\lambda u=0$. For such functions he proves mean value theorems which in the case that Δ^* is the Laplacian operator ($K=0$) reduce to classical theorems (Weber's theorem and, for $\lambda=0$, Gauss's mean value theorem). While, as the author notices, these

generalizations have been observed before [Rothe, Math. Ann. **105**, 672–693 (1931); for the case of $\lambda=0$, even for not necessarily constant K, see Feller, Math. Ann. **102**, 633–649 (1930)], the remaining results of the paper are new. They concern similar mean value theorems for functions which have singularities inside the sphere over which the mean value is taken and are pseudo-harmonic otherwise.

E. H. Rothe (Ann Arbor, Mich.).

Soudan, Robert. Indéformabilité d'un corps homogène à potentiel polyharmonique constant. C. R. Séances Soc. Phys. Hist. Nat. Genève **62**, 87–88 (1945). [MF 16531]

Let V be a homogeneous body yielding the polyharmonic potential

$$U(P)=\delta\int_V v_n(M, P)d\tau_M,$$

where v_n is of the form

$$v_n(M, P)=\sum_{\alpha=-1}^{2n-2} C_\alpha\overline{MP}^\alpha.$$

The author states that it is impossible to deform the body in such a way as to preserve homogeneity with fixed mass without changing U outside the masses. He outlines a method of proof, but no detailed demonstration is given.

F. W. Perkins (Hanover, N. H.).

Poritsky, Hillel. Application of analytic functions to two-dimensional biharmonic analysis. Trans. Amer. Math. Soc. **59**, 248–279 (1946). [MF 15652]

The author discusses solutions of the biharmonic equation

$$(1)\qquad (\partial^2/\partial x^2+\partial^2/\partial y^2)^2u=0$$

for special boundary conditions arising in connection with bending and stretching of plates. As is well known, the general solution of (1) can be expressed in various ways in terms of potential functions or analytic functions, for example, as the real part of $f(z)+z\bar{z}g(z)$, where f and g are arbitrary analytic functions of $z=x+iy$. From a solution having a singularity at a point others can be constructed having the same singularity and satisfying simple boundary conditions along circles or straight lines, if use is made of the "principle of reflection" for analytic functions. Among the cases treated in this manner are the determination of the Green's function of (1) for half-planes and circles, and the determination of Airy's function for those two classes of regions corresponding to given loads at one internal point and to given tractions along the boundary. *F. John.*

Bremekamp, H. On the uniqueness of the solutions of $\Delta^k u=0$. Nederl. Akad. Wetensch., Proc. **48**, 222–228 = Indagationes Math. **7**, 27–33 (1945). (Dutch) [MF 15794]

The starting point is an observation by A. Sommerfeld [Enzyklopaedie Math. Wiss. II A 7c (vol. II.1.1., pp. 504–570)] to the effect that u is uniquely determined in a closed two-dimensional area if $\Delta^k u=0$ and if u and its derivatives normal to a boundary curve up to the order $k-1$ are to assume given values. It is the author's object to provide a proof of this proposition. In doing so the values of u and of the specified normal derivatives are assumed to be zero. The functions u under discussion are assumed to have regular partial derivatives up to the order $2k$ within the given area as well as on the border curve. Special assumptions are introduced with respect to the latter, excluding multiple points and irregularity of tangents. The proof itself consists essentially of repeated applications of Green's theorem. Thereupon the above theorem is extended to a closed three-dimensional volume, introducing suitable restrictions as to its boundary, similar to those in the case of two dimensions. The proof is elaborated along similar lines. It is stated that the case of n dimensions can be handled formally in the same way. *M. J. O. Strutt* (Eindhoven).

Bremekamp, H. Properties of the solutions of $\Delta^k u=0$. Nederl. Akad. Wetensch., Proc. **48**, 229–236 = Indagationes Math. **7**, 34–41 (1945). (Dutch) [MF 15795]

The first property under consideration states that $\Delta^k r^{2k-2}u=0$ if $\Delta u=0$ and if partial derivatives of u up to the order $2k$ exist, r representing the distance from an arbitrary point within the domain of u. It is proved by induction from two subsidiary propositions. The second property states that every solution of $\Delta^k u=0$ may be written as

$$u=r^{2k-2}u_0+r^{2k-4}u_1+\cdots+r^2u_{k-2}+u_{k-1},$$

if $u_0, u_1, \cdots, u_{k-1}$ are harmonic functions. The proof starts from the observation that this property is true if $k=1$ and then proceeds by induction, using the same subsidiary propositions as in the preceding proof. These properties were related to two independent variables. In the case of three independent variables similar properties and proofs are shown to hold. *M. J. O. Strutt* (Eindhoven).

Bottema, O., and Bremekamp, H. On the solutions of the equation $\Delta^\nu u=0$ which satisfy certain boundary conditions. I. Nederl. Akad. Wetensch., Proc. **49**, 424–435 = Indagationes Math. **8**, 279–290 (1946). (Dutch) [MF 16825]

The authors aim at an extension of Poisson's integral formula for potential problems to the problems indicated in the title. The number of independent variables is subsequently taken to be 2, 3 and k. The boundary curve, surface or hypersurface is a circle, a sphere or a hypersphere. On this boundary surface the values of the independent variable u and of its partial derivatives in a direction normal to the surface up to an order ν have given values. The solution is carried out with the aid of a Green's function H of two complete sets of independent variables. If these sets are different, H satisfies the potential equation in question. On the boundary H as well as its normal derivatives up to the order ν are zero. If the two sets of independent variables coincide, either H or its derivatives starting from a definite order have definite singularities. The functions H for a sphere and a circle are determined by trial, making use of the theorems that a solution of the problem in hand is unique [see the second preceding review] and that solutions assume definite simple forms around any point of the region under discussion [see the preceding review]. Elaborate calculations lead up to the required results, the coefficients being determined by solving sets of recurrent relationships. *M. J. O. Strutt* (Eindhoven).

Bottema, O., and Bremekamp, H. On the solutions of the equation $\Delta^\nu u=0$ which satisfy certain boundary conditions. II. Nederl. Akad. Wetensch., Proc. **49**, 436–443 = Indagationes Math. **8**, 291–298 (1946). (Dutch) [MF 16826]

Continuing the argument of the paper reviewed above, the authors elaborate an equivalent of Poisson's integral formula in the case when $\nu=3$, there are two independent variables and the boundary is a circle, along which the solution u and its first and second derivatives have given values. Another derivation of the same formula is given

under the assumption that the given values along the boundary curve may be developed in a Fourier series, the coefficients of which decrease sufficiently rapidly. The authors then proceed to extend their argument to k independent variables in the case $\nu=2$. Equivalents of Poisson's formula are first given in the case where $k\neq 2$, $k\neq 4$. Afterwards the case $k=4$ is dealt with, thus completely solving the problem for $\nu=2$ and any value of k. Some general remarks on the cases of higher values of ν conclude the paper. *M. J. O. Strutt* (Eindhoven).

✶**Carslaw, H. S. Introduction to the Mathematical Theory of the Conduction of Heat in Solids.** Dover Publications, New York, N. Y., 1945. xii+268 pp. $3.50.

Photographic reproduction of the second edition [Macmillan, London, 1921].

Shvetz, M. The course of temperature in twenty-four hours and radiant heat exchange. Bull. Acad. Sci. URSS. Sér. Géograph. Géophys. [Izvestia Akad. Nauk SSSR] 1943, 218–244 (1943). (Russian. English summary) [MF 16628]

In this paper the author studies the course of temperature of air and the underlying surface for twenty-four hours, taking into account the radiant heat exchange and solving, simultaneously, the equations of heat inflow and transfer of radiant energy by the method of successive approximation. *Author's summary.*

Dressel, F. G. The fundamental solution of the parabolic equation. II. Duke Math. J. 13, 61–70 (1946). [MF 15875]

Let

$$(1)\qquad L(u)\equiv \sum_{i,j=1}^{n} a_{ij}\frac{\partial^2 u}{\partial x_i\partial x_j}+\sum_{i=1}^{n} a_i\frac{\partial u}{\partial x_i}+au-\frac{\partial u}{\partial y}=0$$

be the general linear parabolic equation, where a_{ij}, a_i and a are functions of $x_1, \cdots, x_n, y$ and the quadratic form whose coefficients are the $a_{ij}=a_{ji}$ is positive definite if $x=(x_1, \cdots, x_n)$ is in the region R defined by $-\infty<x_i<\infty$ and y in the interval $y'\leqq y\leqq y''$. Under certain differentiability conditions concerning the coefficients of $L(u)$, the existence of a fundamental solution of (1) is proved, that is, a function $\Gamma(x, y; \xi, \eta)$ with the following properties: (i) for $y>\eta$ and each pair x, ξ lying in R, Γ is a regular solution of (1) as a function of x, y; (ii) if T is a subregion of R and $\varphi(x)$ is continuous and bounded in R, then

$$\lim_{y\to\eta}\int_T \varphi(\xi)\Gamma(x, y; \xi, \eta)d\xi$$

equals $\varphi(x)$ if x is an interior point of T and equals 0 if x is an interior point of $R-T$. Under some additional conditions concerning the coefficients of (1), it is shown that Γ considered as a function of ξ, η satisfies the equation adjoint to (1). In a former paper [Duke Math. J. 7, 186–203 (1940); these Rev. 2, 204] the author proved the corresponding facts in the case where R is finite. In order to include the infinite case the estimate of the solution of an integral equation appearing in the construction of Γ had to be refined. The results may also be considered as an extension to arbitrary n of results obtained in the case $n=1$ by Feller [Math. Ann. 113, 113–160 (1936)]. *E. H. Rothe.*

Vranceanu, G. Sur l'équivalence des équations de Laplace. Bull. Math. Soc. Roumaine Sci. 46, 155–180 (1944). [MF 16519]

The author studies the Laplace equation

$$(\mathrm{A})\qquad x_{uv}+ax_u+bx_v+cx=0$$

(a, b, c, functions of u, v) under the transformation

$$(\mathrm{B})\qquad x=\lambda(u, v)x',\quad u=f(u'),\quad v=\phi(v').$$

By starting with the two Darboux relative invariants of (A) a series of absolute invariants is formed by successive logarithmic differentiation in terms of which a necessary and sufficient condition for the equivalence of two equations (A) under transformation (B) can be stated. The case when these invariants depend only on v is developed in some detail, particularly when the first invariant is a polynomial or rational function. Relatively simple equivalence theorems result. Turning to Laplace sequences, the author exhibits a relationship between the invariants of adjoining members. The problem of equivalence of sequences is studied for the special case of the generalized Poisson equation. Here all the invariants of any member of the sequence can be expressed in terms of those of the principal member and conditions for equivalence can be stated in simple form. *J. L. Vanderslice* (College Park, Md.).

Slater, J. C. Physics and the wave equation. Bull. Amer. Math. Soc. 52, 392–400 (1946). [MF 16903]

In this Gibbs lecture the author expounds the thesis that a guiding line of mathematical thought, such as the principle of least action, has often permeated the whole of physics for years, tying together apparently unrelated branches of the subject and at the same time focussing attention on a branch of mathematics and leading to its development. In every case an advance in physics has stimulated some of the most vital and valuable developments in mathematics. At present, the guiding line is the wave equation, both in the classical and the Schrödinger form, and methods developed for quantum-mechanical problems have proved valuable in the classical field. The current need is for further development of general theory and of special methods applicable when the introduction of the appropriate coordinate system renders the wave equation inseparable. *E. T. Copson* (Dundee).

Bureau, Florent. Sur le problème de Cauchy pour les équations aux dérivées partielles, totalement hyperboliques, d'ordre plus grand que 2. C. R. Acad. Sci. Paris 222, 849–851 (1946). [MF 16281]

The equation considered is written in the form $D_\varphi D_\psi u=F(x_1, x_2, x_3)$ with

$$D_\varphi=a_3\frac{\partial^2}{\partial x_3^2}-a_1\frac{\partial^2}{\partial x_1^2}-a_2\frac{\partial^2}{\partial x_2^2},$$

$$D_\psi=b_3\frac{\partial^2}{\partial x_3^2}-b_1\frac{\partial^2}{\partial x_1^2}-b_2\frac{\partial^2}{\partial x_2^2},$$

where the a's and b's are constants such that $0<a_3<a_2<a_1$, $0<b_1<b_2<b_3$. The solution of the Cauchy problem is then given in terms of integrals involving the known function F, the Cauchy data, and a certain function analogous to the "elementary solution" of the Hadamard theory for hyperbolic second order differential equations. *D. C. Lewis.*

Integral Equations

★**Navarro Borrás, F. Conferencias Sobre la Teoria de las Ecuaciones Integrales (Lineales y No-Lineales).** [**Lectures on the Theory of Integral Equations (Linear and Nonlinear)**]. Consejo Superior de Investigaciones Cientificas, Madrid, 1942. 185 pp. (Spanish)

The author gives a concise treatment of the elements of the Fredholm theory of linear integral equations and the theory of linear equations with real symmetric kernels, together with a consideration of nonlinear integral equations; the presentation of the latter topic consists mainly of an account of the principal results of E. Schmidt [Math. Ann. **65**, 370–399 (1908)] and A. Hammerstein [Acta Math. **54**, 117–176 (1930)]. The scope of the monograph is fairly well indicated by the following list of topics of the individual lectures. 1. Introduction; solution of linear equations with degenerate kernels. 2. The resolvent; introduction to the method of Fredholm. 3. Theory of Fredholm (continuation). 4. Theory of Schmidt for nonsymmetric kernels. 5. Theory of Schmidt for symmetric kernels. 6. Existence of proper values and proper functions. 7. Properties of the set of proper values. 8. The theorem of Weyl on the dependence of proper values on the kernel. 9. The Hilbert-Schmidt expansion theorem and solution of the complete equation. 10. Applications of integral equations to the solution of boundary problems for ordinary linear differential equations of the second order. 11. Introduction to the theory of nonlinear integral equations. 12. Method of successive approximations. 13. Introduction to the theory of E. Schmidt for nonlinear equations. 14. The case of bifurcation. 15. The equation of bifurcation. 16. Variation of the solution of the generalized Dirichlet problem with the boundary values. 17. Examples of nonlinear equations which have no solutions. 18. Existence theorems. 19. Amplification of sufficient conditions; uniqueness theorems.

W. T. Reid (Evanston, Ill.).

Lichnerowicz, André. Sur la composition de seconde espèce et les fonctions de Schmidt. C. R. Acad. Sci. Paris **219**, 663–666 (1944). [MF 15299]

If a bounded integrable kernel $L(x, y)$ with singular functions $\{\varphi_i(x), \psi_i(y)\}$ and singular values $\{\lambda_i\}$ can be expressed in the form

$$L(x, y)=\int_a^b H(x, s)K(s, y)ds,$$

where $H(x, s)$ and $K(s, y)$ are bounded integrable kernels with singular functions $\{\varphi_i(x), \chi_i(s)\}$ and $\{\chi_i(s), \psi_i(y)\}$, respectively, the kernel L is said to possess a canonical decomposition; if not, it is said to be prime. The author's main result is that L possesses a canonical decomposition if and only if there exist numbers α_i, β_i such that $\alpha_i\beta_i=\lambda_i$ and the series

$$\sum_{i=1}^{\infty}\varphi_i^2(x)/\alpha_i^2, \quad \sum_{i=1}^{\infty}\psi_i^2(y)/\beta_i^2$$

are convergent. When this is so, one of the two kernels H and K may be taken symmetric. *F. Smithies.*

Azevedo do Amaral, Ignacio M. On the integral equation of the first kind. Anais Acad. Brasil. Ci. **17**, 283–287 (1945). (Portuguese) [MF 15538]

This paper deals with the relation between an integral equation of the first kind $F(x)=\int_a^x\theta(x, \alpha)u(\alpha)d\alpha$ and the first order linear differential equation $dF(x)/dx+M(x)F(x)=u(x)$. *W. T. Reid* (Evanston, Ill.).

Parodi, Maurice. Équations intégrales de Fredholm et calcul symbolique: errata. C. R. Acad. Sci. Paris **222**, 251 (1946). [MF 15995]

Cf. the same C. R. **220**, 870–872 (1945); these Rev. **7**, 206.

Pinney, Edmund. An integral equation occurring in potential theory. Ann. of Math. (2) **47**, 221–232 (1946). [MF 16332]

The integral equation discussed is of the general type

$$\pi^{-1}P\int_C\psi(t)(t-z)^{-1}dt-\lambda(z)\psi(z)=f(z),$$

where C is a path in the complex plane and the integral is a principal value when z is on C. A solution is obtained in the case when $f(z)$ is an analytic function by means of the solution $\phi(z)$ of the corresponding homogeneous equation. The function $\phi(z)$ is actually constructed for the case when C is a chain of line-segments and $\lambda(z)$ takes different constant values on the links of C.

The latter part of the paper applies the results to the solution of two-dimensional electrostatic problems, such as the determination of the field in a condenser formed by two infinitely long parallel coplanar strips. *E. T. Copson.*

Pollaczek, F. Résolution de certaines équations intégrales linéaires de deuxième espèce. J. Math. Pures Appl. (9) **24**, 73–93 (1945). [MF 15969]

The equation studied is

$$\text{(I)} \qquad \phi(x)-(1/2\pi i)\int_{C_a}K(x, \xi)\phi(\xi)d\xi=f(x),$$

where $K=j_0(x)k_0(\xi)/(x-\xi)+\sum_1^q j_i(x)k_i(x)$, C_a is a line of abscissa a, x is to the right of C_a, q is to the left of C_a and j_0, k_0, f/j_0, j_i/j_0 satisfy conditions of the form (II) $|k(\xi)|<c|\xi|^{-\epsilon}$, where $a\leqq\Re(\xi)<a+\delta$; $\epsilon, \delta>0$; and k is analytic, except for n simple poles, for $\Re(\xi)<a+\delta$. A solution is obtained, with the aid of a finite number of integrations, for $|z|$ small. The methods involved in solving (I) are also effectively used to treat a number of problems similar to (I), including a system of importance in some of the author's work in probability. *W. J. Trjitzinsky* (Urbana, Ill.).

Kveselava, D. Singular integral equations with discontinuous coefficients. Trav. Inst. Math. Tbilisi [Trudy Tbiliss. Mat. Inst.] **13**, 1–27 (1944). (Georgian. Russian summary) [MF 14618]

The equation studied is

$$\alpha(t_0)\phi(t_0)+\pi^{-1}\beta(t_0)\int(t-t_0)^{-1}\phi(t)dt+\pi^{-1}\int\gamma(t_0, t)\phi(t)dt=f(t_0),$$

where integration is over a smooth simple closed contour L and α, β, γ, f are given functions of Hölder class on L (except for a finite number of discontinuities of the first kind). Such equations are solved on the basis of a Riemann boundary value problem (R) with discontinuous coefficients. With the aid of properties of Cauchy integrals the problem (R) is solved directly, without reduction to a discontinuous problem. Among the theorems are some of the F. Noether type, proved much more simply than heretofore, as well as results relating to regularization and equivalence.

W. J. Trjitzinsky (Urbana, Ill.).

Harazov, D. F. On a class of singular integral equations whose kernels are meromorphic functions of a parameter. Trav. Inst. Math. Tbilissi [Trudy Tbiliss. Mat. Inst.] 13, 139–152 (1944). (Russian. Georgian summary) [MF 14623]

The equation studied is

$$(1)\qquad a(x)\phi(x)-\int H(x,t;\lambda)(t-x)^{-1}\phi(t)dt=f(x),$$

where integration is in the sense of Cauchy principal values over a simple closed contour L; x, t are on L; a, f, H are of Hölder type on L in x and t; H is meromorphic in λ, of form $G_1(x,t;\lambda)/G_0(\lambda)$, where G_1, G_0 are entire in λ; if $G_0(\lambda_0)=0$, then $G_1(x,t;\lambda_0)\neq 0$ for all x, t on L. Equation (1) is rewritten in the form

$$(\mathrm{A})\qquad A_\lambda\phi=a(x)\phi(x)-\beta(x;\lambda)\int\phi(t)(t-x)^{-1}dt \\ -\int K(x,t;\lambda)\phi(t)dt=f(x),$$

where

$$\beta(x;\lambda)=H(x,x;\lambda),\ K(x,t;\lambda)(t-x)=H(x,t;\lambda)-H(x,x;\lambda).$$

Let D denote the set of λ for which $a^2(x)+\pi^2\beta^2(x,\lambda)=0$, even if for one x on L. The complement of D consists of a denumerable infinity (at most) of connected open sets Λ_j. It is assumed that the origin is interior to L;

$$n(\lambda)=\tfrac{1}{2}\pi^{-1}[\arg(a'/a'')]_L,$$

where $a'=a+i\pi\beta$, $a''=a-i\pi\beta$, is the index of (A). The fundamental result is this: in every Λ_j and for all $\lambda\epsilon\Lambda_j-R_j$ (R_j a set with no limiting points in Λ_j) the numbers of distinct solutions of $A_\lambda\phi=0$ and of the associated equation $A_\lambda'\psi=0$ have constant values, say v_j and v_j', with v_j-v_j' equal to the index of ($A_\lambda\phi=0$) in Λ_j. This work has connections with a paper by G. Giraud [Ann. Sci. École Norm. Sup. (3) 56, 119–172 (1939); these Rev. 1, 145].

W. J. Trjitzinsky (Urbana, Ill.).

Cinquini, Silvio. Sopra i problemi di valori al contorno per equazioni integro-differenziali. Ann. Mat. Pura Appl. (4) 20, 257–270 (1941). [MF 16605]

The principal result of this paper is that the integro-differential equation

$$y'(x)=y'(x_0)+\int_{x_0}^{x}f\left\{x,y(x),y'(x);\int_{x_0}^{x}g(x,z,y(z),y'(z))dz\right\}dx$$

has at least one solution $y(x)$, with $y(x)$ and $y'(x)$ absolutely continuous on $a\leq x\leq b$, satisfying the boundary conditions $y(x_1)=y_1$, $y(x_2)=y_2$ with $a\leq x_1<x_2\leq b$, provided: (a) $f(x,y,y',u)$ is continuous in x on (a,b) for each fixed y, y', u, and continuous for all y, y', u for each fixed x on (a,b); (b) $g(x,z,y,y',u)$ has similar continuity properties with the interval $a\leq x\leq b$ replaced by the triangle T_0: $a\leq x\leq b$; $a\leq z\leq x$; (c) there exist two positive integrable functions $\psi(x)$ and $\psi_0(z)$ so that $|f(x,y,y',u)|\leq\psi(x)$ and $|g(x,z,y,y')|\leq\psi_0(z)$ for all values in the ranges of definition of f and g. By modifying the dominance conditions to which f and g are subjected, a number of modifications of this theorem are derived.

T. H. Hildebrandt.

Functional Analysis, Ergodic Theory

Monteiro, António, et Ribeiro, Hugo. La notion de fonction continue. Summa Brasil. Math. 1, 1–8 (1945). (French. Portuguese summary) [MF 15865]

Given a partially ordered set P and an operation k on P into itself, one calls any other such operation f continuous if, for $A\epsilon P$, $f(k(A))\subset k(f(A))$. The derivability or not of various formulae of familiar appearance but new context from various axioms on (P,k) is considered.

R. Arens.

Braconnier, Jean. Sur les espaces vectoriels localement compacts. C. R. Acad. Sci. Paris 222, 777–778 (1946). [MF 16168]

Theorems about a topological vector space V over a locally compact nondiscrete field K, including: if V is complete then V is locally compact if and only if V has finite dimension n, in which case $V\cong K^n$; K need not be commutative.

W. Ambrose (Ann Arbor, Mich.).

Judin, A. Sur le complètement des ensembles semi-ordonnés. Leningrad State Univ. Annals [Uchenye Zapiski] 83 [Math. Ser. 12], 57–61 (1941). (Russian. French summary) [MF 16488]

The author considers partially ordered linear spaces in which the ordering satisfies the following axioms: (I) $y>0$ implies $y\neq 0$; (II) $y_1>0$ and $y_2>0$ imply $y_1+y_2>0$; (III) for any y, there exists a $\bar{y}$ such that $\bar{y}-y>0$; (III$_1$) for any y, there exists sup $(0,y)$. A further axiom is considered: (V) for any set E with an upper bound, there exists a least upper bound. The author proves that a linear space satisfying axioms (I), (II), (III), and (III$_1$) can be imbedded in a space satisfying (I), (II), (III), and (V) if and only if no negative element is the limit, in a certain sense, of positive elements.

E. Hewitt (Bryn Mawr, Pa.).

Toranzos, Fausto I. On projectivity in Hilbert spaces. Publ. Inst. Mat. Univ. Nac. Litoral 7, 189–197 (1945). (Spanish. English summary) [MF 14580]

[A more accurate title would be "On projective transformations in Hilbert space."] This paper is based on work of Vitali, who considered projective geometry in a space obtained by adjoining to Hilbert space a hyperplane of points at infinity [Ann. Mat. Pura Appl. (4) 11, 155–179 (1932)]. Using the results of Vitali on the factorization of the general projective transformation, the author proves that, in a suitable coordinate system, the equation of a hyperquadric is of the second degree. A hyperquadric is defined as the transform of a hypersphere by a projective transformation.

E. R. Lorch (New York, N. Y.).

Julia, Gaston. Sur les racines carrées hermitiennes d'un opérateur hermitien positif donné. C. R. Acad. Sci. Paris 222, 707–709 (1946). [MF 16170]

The problem is to determine all bounded Hermitian square roots of a given bounded positive definite operator K on Hilbert space. Let H be the unique positive definite square root of K, $\mathfrak{M}$ the closed linear manifold spanned by the range of K, V_1 a closed linear submanifold of $\mathfrak{M}$ invariant under K, and $V_2=M\ominus V_1$; a necessary and sufficient condition that h be a Hermitian square root of K is that it have the form $h=(P_{V_1}-P_{V_2})H$, where P_V is the orthogonal projection on the manifold V. Write $H_i=P_{V_i}H$ and let $E(\lambda)$ and $E_i(\lambda)$ be the spectral families of H and H_i, respectively; then $E_i=I-P_{V_i}+P_{V_i}E$. The spectral family $\mathfrak{E}(\lambda)$ of h is

given by $P_{V_1}(I-E(-\lambda-0))$ for $\lambda<0$ and $I-P_{V_1}+P_{V_1}E(\lambda)$ for $\lambda\geqq 0$. *P. R. Halmos* (Chicago, Ill.).

Julia, Gaston. Remarques sur les racines carrées hermitiennes d'un opérateur hermitien positif borné. C. R. Acad. Sci. Paris **222**, 829–832 (1946). [MF 16275]

Since (in the notation of the preceding review) $h=H_1-H_2$, it follows that h is determined by the two spectral families E_1 and E_2. The family E_1 is such that (1) $E\leqq E_1$ and (2) E_1-E is a decreasing function of λ; conversely, if E_1 satisfies (1) and (2) and E_2 is defined by $E_2=E+I-E_1$ then the corresponding h is a Hermitian square root of K. The author remarks that the obvious procedure of constructing indefinite square roots of K (changing the sign of the integrand in the spectral representation of H on a subset of the spectrum of H) is a special case of his procedure and does not yield all square roots of K but only those which commute with every operator which commutes with K.

P. R. Halmos (Chicago, Ill.).

Julia, Gaston. Sur la représentation spectrale des racines hermitiennes d'un opérateur hermitien positif donné. C. R. Acad. Sci. Paris **222**, 1019–1022 (1946). [MF 16378]

The author reduces the problem of finding the most general Hermitian square root of a given positive definite operator K on Hilbert space to the cases where K has either pure point spectrum or else pure continuous spectrum and, in these cases, obtains expressions for the spectral family of the square root. *P. R. Halmos* (Chicago, Ill.).

Julia, Gaston. S ır les racines carrées self-adjointes d'un opérateur self-adjoint positif non borné. C. R. Acad. Sci. Paris **222**, 1061–1063 (1946). [MF 16520]

The author reduces the problem of finding all self-adjoint square roots of an unbounded, self-adjoint, positive operator on Hilbert space to the corresponding problem for bounded operators [cf. the three preceding reviews].

P. R. Halmos (Chicago, Ill.).

Julia, Gaston. Sur les racines $n^{\text{ièmes}}$ hermitiennes d'un opérateur hermitien donné. C. R. Acad. Sci. Paris **222**, 1465–1468 (1946).

Using the methods of his investigations of square roots [cf. the four preceding reviews] the author studies nth roots of Hermitian operators. He proves that (a) if K is Hermitian and n is odd then K has exactly one nth root which has the same domain as K and is definite or indefinite along with K, (b) if K is positive definite and n is even then K has infinitely many nth roots and (c) if K is negative definite and n is even then K has no nth roots.

P. R. Halmos (Chicago, Ill.).

Lifshitz, I. M. On the theory of regular perturbations. C. R. (Doklady) Acad. Sci. URSS (N.S.) **48**, 79–81 (1945). [MF 15220]

Let L be a semibounded Hermitian operator whose discrete spectrum $\lambda_1, \lambda_2, \cdots$ and corresponding normed eigenfunctions $\psi_1, \psi_2, \cdots$ are supposed to be known. Let Λ be another Hermitian operator and t a real parameter. The author sketches a perturbation theory for the operator $L+t\Lambda$ which holds not only if the perturbation $t\Lambda$ is small enough but under conditions expressing that the quantities $\Lambda_{pq}/(\lambda_p\lambda_q)^{\frac{1}{2}}$, where $\Lambda_{pq}=(\Lambda\psi_p, \psi_q)$, decrease rapidly enough. The condition mainly treated is that $\sum|\Lambda_{pq}|^2/|\lambda_p\lambda_q|$ converges (more general conditions are mentioned). Under this condition an equation for the eigenvalues of the perturbed operator is set up and an expression for the corresponding eigenfunctions is given. Some remarks concerning oscillation properties of the eigenvalues are added.

E. H. Rothe (Ann Arbor, Mich.).

Eberlein, William F. A note on the spectral theorem. Bull. Amer. Math. Soc. **52**, 328–331 (1946). [MF 16200]

The spectral resolution for bounded self adjoint operators is obtained by considering $(p(H)f, g)$ as a linear functional on the p in C. The Riesz theorem on the representation of such functionals yields a $\rho(\lambda, f, g)$ of bounded variation which is shown to be equal to $(F(\lambda)f, g)$. *F. J. Murray.*

Please also see Errata p. 621.

Zaanen, A. C. Ueber vollstetige symmetrische und symmetrisierbare Operatoren. Nieuw Arch. Wiskunde (2) **22**, 57–80 (1943). [MF 15700]

For the validity of many theorems concerning linear operators in Hilbert space the separability and, in the case of completely continuous symmetric operators, also the completeness of the space are not necessary. This was shown by Rellich [Math. Ann. **110**, 342–356 (1934)], who used mainly variational methods. In the present paper the iteration method introduced by Kellogg [Math. Ann. **86**, 14–17 (1922)] for the treatment of linear integral equations is used throughout. The author starts by proving anew with this method some of Rellich's results concerning the existence of eigenelements, their extremum properties, etc. In the sequel related topics are treated; in particular, the corresponding theorems are proved concerning symmetrizable operators, that is, linear operators of the form HK, where H is positive definite, symmetric and completely continuous while K is linear. Moreover, the author constructs a new space such that to each symmetrizable operator in the original space there corresponds a symmetric operator in the new one. Questions of completing the space and continuation of the operator to this completed space are also treated. Finally, operators are treated which are strongly completely continuous in the sense of Rellich [Math. Ann. **111**, 560–567 (1935)]. For these a theorem is proved which is the generalization of the classical expansion theorem according to eigenfunctions in the theory of integral equations. *E. H. Rothe* (Ann Arbor, Mich.).

Nakano, Hidegorô. Über Struktur von Spektren im allgemeinen Euklidischen Raum. Proc. Phys.-Math. Soc. Japan (3) **23**, 871–882 (1941). [MF 15010]

This paper continues earlier work of the author [cf. Ann. of Math. (2) **42**, 657–664 (1941); Math. Ann. **118**, 112–113 (1941); these Rev. **3**, 51; **4**, 13] on the structure of Abelian systems of projections in nonseparable Hilbert spaces. A projection-valued operator $E(Z)$ is said to be of simple type if there is a system of sets Z with a fixed subideal with the same relative intersection and joining properties. The paper is concerned with the resolution of a general system relative to operators of a simple type. Sufficient conditions for such a resolution are obtained. *F. J. Murray.*

Nakano, Hidegorô. Über Einführung der teilweisen Ordnung im reellen Hilbertschen Raum. Proc. Phys.-Math. Soc. Japan (3) **26**, 1–8 (1944). [MF 15080]

This paper is concerned with converse questions arising from earlier work of the author in which a partially ordered set was represented in real Hilbert space by a type of operational calculus. A simple Abelian system of projections is given and the corresponding partial ordering is obtained.

Part of this earlier work was not available to the reviewer. [The references are: J. Fac. Sci. Imp. Univ. Tokyo 4, 201–382 (1942), in particular, p. 361; see also Proc. Imp. Acad. Tokyo **17**, 311–317 (1941); Jap. J. Math. **17**, 425–511 (1941), in particular, p. 436; these Rev. **3**, 210; **7**, 249.]
F. J. Murray (New York, N. Y.).

Arnous, Edmond. Sur les représentations unitaires des groupes abéliens localement compacts dans l'espace de Hilbert. Une extension d'un théorème de M. H. Stone. C. R. Acad. Sci. Paris **222**, 215–217 (1946). [MF 15993]

Further announcement of results obtained by Neumark [Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **7**, 237–244 (1943); these Rev. **5**, 272], Ambrose [Duke Math. J. **11**, 589–595 (1944); these Rev. **6**, 131] and Godement [same C. R. **218**, 901–903 (1944); these Rev. **7**, 307]. *M. H. Stone* (Chicago, Ill.).

Godement, Roger. Sur les partitions finies des fonctions de type positif. C. R. Acad. Sci. Paris **222**, 36–37 (1946). [MF 15976]

If U_x is a unitary representation of a locally compact group G on a Hilbert space $\mathfrak{H}$, then $\{\mathfrak{H}, U_x\}$ is called a unitary structure for G. In a natural way, notions of isomorphism ($\sim$), direct product ($\times$), and inclusion ($>$) are defined for structures. Let f be an element of the class $\mathfrak{T}$ of functions of positive type on G and denote by $\{\mathfrak{H}_f, U_x\}$ the associated unitary structure for G, where $\mathfrak{H}_f$ is realized in the space $\mathfrak{C}$ of all bounded continuous functions on G [Gelfand and Raikov, Rec. Math. [Mat. Sbornik] N.S. **13**(55), 301–316 (1943); Godement, same C. R. **221**, 69–71 (1945); these Rev. **6**, 147; **7**, 254]. If $g-f\epsilon\mathfrak{T}$, write $f<g$. Then $f<g$ implies $\{\mathfrak{H}_f, U_x\}<\{\mathfrak{H}_g, U_x\}$ and $\mathfrak{H}_f\cap\mathfrak{H}_g=\{0\}$ if, and only if, $\inf(f, g)=0$. Consider the partition $f=f_1+\cdots+f_n$ in $\mathfrak{T}$. Then $\mathfrak{H}_f=\mathfrak{H}_{f_1}+\cdots+\mathfrak{H}_{f_n}$ in $\mathfrak{C}$ and $\{\mathfrak{H}_f, U_x\}<\{\mathfrak{H}_{f_1}, U_x\}\times\cdots\times\{\mathfrak{H}_{f_n}, U_x\}$. If the $\mathfrak{H}_{f_i}$ are linearly independent in $\mathfrak{C}$, then $<$ can be replaced by $\sim$, in which case the partition is said to be essential. If $\mathfrak{H}_f$ contains no invariant subspaces, then f is said to be irreducible. The element f is irreducible if, and only if, it admits no essential partitions. *C. E. Rickart* (New Haven, Conn.).

Godement, Roger. Sur certains opérateurs définis dans l'espace d'une fonction de type positif. C. R. Acad. Sci. Paris **222**, 213–215 (1946). [MF 15992]

Let $\{\mathfrak{H}_f, U_x\}$ denote the unitary structure determined by an element $f\epsilon\mathfrak{T}$ [see the preceding review]. Denote by $\mathfrak{R}$ the class of all $f-g$ where $f, g\epsilon\mathfrak{T}$. If $f\epsilon\mathfrak{T}$, then g is said to be f-bounded below (above) provided there exists n such that $-nf<g$ ($g<nf$). Set $\mathfrak{T}_f=\mathfrak{T}\cap\mathfrak{H}_f$ and $\mathfrak{R}_f=\mathfrak{R}\cap\mathfrak{H}_f$. To each $g\epsilon\mathfrak{R}_f$, which is f-bounded above or below, there corresponds a self-adjoint operator A_g which commutes with U_x and for which $g<h$ implies $A_g<A_h$. A study of these operators leads to the following results. (1) If g, h are f-bounded then $k(x)=(g, h_x)$, where $h_x(t)=h(x^{-1}t)$, belongs to $\mathfrak{R}_f$ if, and only if, $k(x)=\overline{k(x^{-1})}$. (2) If g is f-bounded below there exists a "spectral decomposition" of g of the form $g=\int_n^\infty\lambda de_\lambda$, where $f=\int_n^\infty de_\lambda$, $0<e_\lambda<e_\mu<f$ ($\lambda<\mu$), $(e_\lambda, U_xe_\mu)=(e_\mu, U_xe_\lambda)$, and $\lim_{\mu\to\lambda+}e_\mu=e_\lambda$ in $\mathfrak{H}_f$. (3) Every $g\epsilon\mathfrak{T}_f$ is the limit in $\mathfrak{H}_f$ of an increasing sequence of f-bounded elements of $\mathfrak{T}_f$ and possesses a unique square root $k\epsilon\mathfrak{T}_f$ with $g(x)=(k, k_x)$. (5) If g is f-bounded then $g=g^+-g^-$, where $g^+, g^-\epsilon\mathfrak{T}_f$ and $\inf\{g^+, g^-\}=0$. *C. E. Rickart* (New Haven, Conn.).

Godement, Roger. Sur quelques propriétés des fonctions de type positif définies sur un groupe quelconque. C. R. Acad. Sci. Paris **222**, 529–531 (1946). [MF 16032]

Denote by $\mathfrak{L}$ the class of continuous functions on G which vanish outside a compact set and let $\mathfrak{P}$ be the class of all ϕ such that

$$\int\int\phi(x^{-1}y)f(x)\overline{f(y)}dxdy\geqq 0$$

for $f\epsilon\mathfrak{L}$. Define $\phi<\psi$ provided $\psi-\phi\epsilon\mathfrak{P}$ and let $\mathfrak{T}$ be the class of continuous elements of $\mathfrak{P}$. An element $\phi\epsilon\mathfrak{T}$, such that $\mathfrak{H}_\phi$ [see the preceding two reviews] contains no invariant subspaces, is said to be elementary and finite linear combinations of elementary functions are called trigonometric polynomials. Let $\mathfrak{P}^2=\mathfrak{P}\cap L^2$, $\mathfrak{T}^2=\mathfrak{T}\cap L^2$ and $\phi*\psi(x)=\int\phi(xy)\overline{\psi(y)}dy$, where L^2 is a Lebesgue space on G and integration is with respect to the left invariant Haar measure. The author states the following results. (1) Every continuous function is the limit, uniformly on compact sets, of trigonometric polynomials and if μ is a finite Radon measure such that $\int\phi(x)d\mu(x)=0$ for every elementary $\phi\epsilon\mathfrak{T}$ then $\mu=0$. (2) If $\phi\epsilon\mathfrak{P}$ is bounded in a neighborhood of the identity then ϕ is a. e. equal to a continuous function. (3) If f is bounded, continuous and belongs to L^1 there exists at least one nonzero elementary function which is the limit, uniformly on compact sets, of linear combinations of translations of f. This result was obtained by A. Beurling [Acta Math. **77**, 127–136 (1945); these Rev. **7**, 61] for the case of the real line without the L^1 restriction. (4) $\mathfrak{T}^2$ is dense in $\mathfrak{P}^2$. (5) Every $\phi\epsilon\mathfrak{T}^2$ is of the form $\psi*\psi$. (6) Every $\phi\epsilon\mathfrak{P}^2$ has a spectral decomposition $\phi=\int_0^\infty\lambda^{-1}de_\lambda$, where $e_\lambda\epsilon\mathfrak{T}^2$, $e_\lambda*e_\mu=e_\mu*e_\lambda=e_\lambda$ and $e_\lambda<e_\mu$ for $\lambda<\mu$. *C. E. Rickart.*

Orihara, Masae. Sur les anneaux des opérateurs. I. Proc. Imp. Acad. Tokyo **20**, 399–405 (1944). [MF 14903]

Orihara, Masae. Sur les anneaux des opérateurs. II. Proc. Imp. Acad. Tokyo **20**, 545–553 (1944). [MF 14922]

The author introduces a new approach to the trace theory, based on the Hahn-Banach extension theorem. The notion of unitary equivalence for definite operators is used to define the function p upon which this extension depends. This treatment does not involve the separability of the space or the factor character of the ring, but only its "finiteness." There are, however, certain basic existence questions which the author ignores. The connection with the original dimensionality considerations of Murray and von Neumann is given in the second paper. There is also a discussion which claims to represent a given ring in terms of "maximal ideals." *F. J. Murray* (New York, N. Y.).

Kondô, Motokiti. Les anneaux des opérateurs et les dimensions. Proc. Imp. Acad. Tokyo **20**, 389–398 (1944). [MF 14902]

Kondô, Motokiti. Sur les sommes directes des espaces linéaires. Proc. Imp. Acad. Tokyo **20**, 425–431 (1944). [MF 14908]

Kondô, Motokiti. Sur la réductibilité des anneaux des opérateurs. Proc. Imp. Acad. Tokyo **20**, 432–438 (1944). [MF 14909]

Kondô, Motokiti. Les anneaux des opérateurs et les dimensions. II. Proc. Imp. Acad. Tokyo **20**, 689–693 (1944). [MF 14943]

These papers consider rings of operators on general spaces in a formal manner. In the first, relations between M, M',

M'', M''', etc. are obtained and a dimensionality theory is developed. However, the reviewer believes that the more interesting aspects of dimensionality have been avoided by the author's definition of equality. The remaining papers consider the reducibility of rings of operators but again the factor character of the terms in the resulting sum is avoided.
F. J. Murray (New York, N. Y.).

Schatten, Robert. The cross-space of linear transformations. Ann. of Math. (2) **47**, 73–84 (1946). [MF 15660]

The author shows that the associate of the greatest crossnorm is the least crossnorm whose associate is also a crossnorm. Furthermore, it is shown that the conjugate $(B_1 \otimes_\gamma B_2)^*$ of the space obtained by using the greatest crossnorm is the set of all linear transformations from B_1 to B_2^*. A transformation is degenerate if its range is finite dimensional. The relationship of the degenerate transformations to the set of all linear transformations is discussed. Special results for reflexive spaces and for Hilbert space are obtained.
F. J. Murray (New York, N. Y.).

Dunford, Nelson, and Schatten, Robert. On the associate and conjugate space for the direct product of Banach spaces. Trans. Amer. Math. Soc. **59**, 430–436 (1946). [MF 16465]

This paper presents examples in which the conjugate space of the direct product of two Banach spaces is not the direct product by the obvious norm of the conjugates of the original spaces. One example involves the direct product of L by itself and a second example uses l. For a suitable norm the direct product of L by itself is the analogous space $L_{(2)}$ for the unit square. Thus the conjugate spaces are spaces of essentially bounded measurable functions, M and $M_{(2)}$, and it is shown that $M_{(2)}$ contains an element which cannot be approximated by the "expressions" whose closure constitutes $M \otimes M$. *F. J. Murray* (New York, N. Y.).

Mackey, George W. Note on a theorem of Murray. Bull. Amer. Math. Soc. **52**, 322–325 (1946). [MF 16198]

If M is a closed linear subspace of a separable normed linear space B there is a closed linear subspace N of B such that $MN=0$ and $M\dotplus N$ is dense in B. This generalizes a result of F. J. Murray [Trans. Amer. Math. Soc. **58**, 77–95 (1945); these Rev. **7**, 124]. *N. Dunford.*

Lévy, Paul. Sur les fonctionnelles bilinéaires. C. R. Acad. Sci. Paris **222**, 125–127 (1946). [MF 15985]

Let ϕ be a continuous linear functional on the Banach space X of all continuous functions of two variables defined on the square $0\leqq s\leqq 1$, $0\leqq t\leqq 1$. Then, if x and y are members of the Banach space C of all continuous functions of one variable defined on the interval $[0, 1]$, the function $\psi(x, y)=\phi(x(s)y(t))$ is a continuous bilinear functional on $C\otimes C$. The author shows by an example that not every continuous bilinear functional on $C\otimes C$ may be so derived from a continuous linear functional on X.
G. W. Mackey (Cambridge, Mass.).

Monna, A. F. On a linear P-adic space. Nederl. Akad. Wetensch. Verslagen, Afd. Natuurkunde **52**, 74–82 (1943). (Dutch. German, English and French summaries) [MF 15937]

For a fixed prime integer P let K denote the field of P-adic numbers and let $|a|$ denote P-adic absolute value. If p is a real number not less than 1, define $l_p^{(p)}$ to consist of all P-adic sequences $x=\{x_i\}$ for which the norm $\|x\|=\{\sum_{i=1}^\infty |x_i|^p\}^{1/p}$ is finite. With respect to this norm $l_p^{(p)}$ is complete; it is shown to have various properties analogous to those of ordinary Banach space. A linear operator U is continuous if and only if there exists a real number M such that, for every $a\epsilon K$, $\|U(x)\|\leqq M|a|$ whenever $\|x\|\leqq|a|$, and similarly for linear functionals. The continuous linear functionals f are those and only those of the form $f(x)=\sum_{i=1}^\infty c_i x_i$, where $\{c_i\}$ is a bounded P-adic sequence; moreover, $\|f\|=\max\{|c_i|\}$. Weak convergence is discussed for functionals and for elements of $l_p^{(p)}$; in the latter case it is conjectured to be equivalent to ordinary convergence [see the following review]. *I. S. Cohen.*

Monna, A. F. On weak and strong convergence in a P-adic Banach space. Nederl. Akad. Wetensch. Verslagen, Afd. Natuurkunde **52**, 207–211 (1943). (Dutch. German, English and French summaries) [MF 15939]

The conjecture mentioned in the preceding review is shown to be false. It is even possible for a sequence of elements with unbounded norms to converge weakly to zero, contrary to the situation in ordinary Banach space.
I. S. Cohen (Philadelphia, Pa.).

Kunisawa, Kiyonori. Integrations in a Banach space. Proc. Phys.-Math. Soc. Japan (3) **25**, 524–529 (1943). [MF 15067]

The author gives a simple reformulation of the reviewer's theory of integration for functions with values in a Banach space [Trans. Amer. Math. Soc. **38**, 357–378 (1935)] in the case of a domain of finite total measure. He shows that a bounded function is integrable if and only if it admits finite partitions into measurable sets S_i with arbitrarily small "oscillation" diam $(\sum_i m(S_i)x(S_i))$, and that an unbounded function x is integrable if and only if it is a "limit" $x=\lim x_n$ of bounded integrable functions x_n, in the sense that $\lim_{n\to\infty} I(x_n, S)$ exists for all S (that is, the "relative integral" converges on each measurable subset). *G. Birkhoff.*

Macphail, M. S. Integration of functions in a Banach space. Nat. Math. Mag. **20**, 69–78 (1945). [MF 15411]

This paper is a simplification of R. L. Jeffery's presentation of G. Birkhoff's integral [Duke Math. J. **6**, 706–718 (1940); these Rev. **2**, 103]. Following Jeffery, the author defines the integral, first for bounded functions, then for unbounded functions, by a limiting process. The simplification consists in replacing, in the definition of the integral for bounded functions, infinite partitions into measurable subsets by finite ones; otherwise, the development follows Jeffery's paper closely. An outline of a similar treatment of G. B. Price's integral [Trans. Amer. Math. Soc. **47**, 1–50 (1940); these Rev. **1**, 239] is also given at the end of the paper. *J. Dieudonné* (São Paulo).

Maslow, A. A note on Birkhoff's product-integral. Leningrad State Univ. Annals [Uchenye Zapiski] **83** [Math. Ser. **12**], 42–56 (1941). (Russian. English summary) [MF 16487]

Let $X(t)$ be a function whose values are "displacements" in a Banach space R; assume that, for $a\leqq t\leqq b$, $X(t)$ is in the class called (AL) by G. Birkhoff [J. Math. Phys. Mass. Inst. Tech. **16**, 104–132 (1937)]. This paper shows the equivalence of three approaches to the definition of a product integral of $X(t)$. The first of these is the constructive definition of the "Lebesgue" product integral defined in the preceding reference; this leads to a product integral $I(X, [\alpha, \beta])$

of $X(t)$ from α to β, $a \leq \alpha \leq \beta \leq b$. The second definition uses three properties which characterize $I(X, [\alpha, \beta])$ as axioms. The third definition begins with an integral equation

$$(*) \qquad z(y, t) = \int_a^t \{X(t)z(y, t) - z(y, t)\} dt + y,$$

where y and $z(y, t)$ are in R, $a<t<b$, and the integral is taken in the sense of Birkhoff [Trans. Amer. Math. Soc. 38, 357–378 (1935)]. After proving existence and uniqueness conditions for $(*)$ it is shown that $A(t)$ is a product integral of $X(t)$ if and only if $z(y, t) = A(t)y$ is a solution of $(*)$.
M. M. Day (Urbana, Ill.).

Michal, A. D. The Fréchet differentials of regular power series in normed linear spaces. Duke Math. J. 13, 57–59 (1946). [MF 15874]

If E_1 and E_2 are Banach spaces, a homogeneous polynomial of degree n on E_1 to E_2 is a function $p_n(x)$ with values in E_2, defined for all elements x in E_1 and having the properties (a) $p_n(tx) = t^n p_n(x)$, (b) $p_n(x+ty)$ is a polynomial of degree not greater than n in the numerical variable t, with coefficients in E_2, (c) $\|p_n(x)\| \leq m\|x\|^n$ for some constant m and every x; the smallest m satisfying (c) is called the modulus $m(p_n)$ of the homogeneous polynomial. A series of the form $f(x) = \sum_0^\infty p_n(x)$ is called a power series. The author considers real Banach spaces only, and defines the radius of analyticity r of the power series as the radius of convergence of the ordinary power series $\sum_0^\infty m(p_n)t^n$. He proves that if $r>0$ the function $f(x)$ has Fréchet differentials of all orders when $\|x\|<r$ and that these differentials are given by successive term-by-term differentiation of the series for $f(x)$. For complex Banach spaces this result is well known. It was first proved by R. S. Martin [California Institute of Technology thesis, 1932]. The proof for real Banach spaces, here given for the first time, is very simple. The key to the proof is a weak form of Markoff's inequality for the derivative of a polynomial [cf. A. C. Schaeffer, Bull. Amer. Math. Soc. 47, 565–579 (1941); these Rev. 3, 111]. *A. E. Taylor* (Los Angeles, Calif.).

Sanvisens, Francisco. The indicatrices of functionals of analytic n-vectors and their application to the integration of rational functions. Publ. Inst. Mat. Univ. Nac. Litoral 6, 225–235 (1946). (Spanish)

The paper also appeared in Revista Mat. Hisp.-Amer. (4) 4, 60–70 (1944); these Rev. 6, 261.

Kawada, Yukiyosi. Über einen schwachen Ergodensatz. Proc. Imp. Acad. Tokyo 18, 343–349 (1942). [MF 14766]

Notations and definitions: $\mathfrak{B}$ is a Banach space, $\overline{\mathfrak{B}}$ its adjoint, $\mathfrak{G}$ is a group; T_x is an isometry of $\mathfrak{B}$ with itself for $x \epsilon \mathfrak{G}$; $T_x \cdot T_y = T_{xy}$; the set of $T_x f$ with f fixed and x variable is conditionally compact in $\mathfrak{B}$; $\mathfrak{N}(f)$ is the closed linear manifold spanned by $T_x f$, $x \epsilon \mathfrak{G}$; U is a linear operator of $\overline{\mathfrak{B}}$ on itself. Condition (B) on U is that, if for a given $L_0 \epsilon \overline{\mathfrak{B}}$ and $f_0 \epsilon \mathfrak{B}$ the set of all f such that $L_0 f = 0$ includes $\mathfrak{N}(f_0)$, in other words, $(f; L_0(f) = 0) \supset \mathfrak{N}(f_0)$, then $(f; UL_0(f) = 0) \supset \mathfrak{N}(f_0)$. The operator U is strongly positive if for every $f \epsilon \mathfrak{B}$ there is a $K(f) \geq 0$ such that, for all $L \epsilon \overline{\mathfrak{B}}$,

$$\Re(UL(f)) \geq K(f) \cdot \text{g.l.b.}_x \{\Re(L(T_x f))\},$$

where $\Re$ denotes the real part. Let $\|U\|_{\mathfrak{G}} = \text{l.u.b.} \; |(UL)f|$ for $|L(T_x f)| \leq 1$. The principal result is the following theorem: if U has property (B), is strongly positive and satisfies $\|U\|_{\mathfrak{G}} \leq 1$, then for every $L \epsilon \overline{\mathfrak{B}}$ there is an $L_0 \epsilon \overline{\mathfrak{B}}$ such that $\lim_{n\to\infty} n^{-1}(L + UL + U^2L + \cdots + U^{n-1}L) = L_0$, the convergence being the weak convergence as a functional (that is, convergence holds when the functionals in question are applied to any $f \epsilon \mathfrak{B}$). *E. R. Lorch* (New York, N. Y.).

Theory of Probability

***Kerrich, J. E. An Experimental Introduction to the Theory of Probability.** Einar Munksgaard, Copenhagen, 1946. 98 pp. 8.50 Kroner.

The fundamental notions of probability up to the theorem on compound probabilities (actually the definition of conditional probability) are developed starting from a commonsense analysis of experimental data of 10,000 spins of a coin and similar experiments with an unsymmetric coin and with urns. The book leads up to the binomial distribution; its approach to the normal distribution is illustrated graphically.
W. Feller (Ithaca, N. Y.).

v. Mises, R. On the probabilities in a set of games and the foundation of probability theory. Revista Ci., Lima 47, 435–456 (1945). [MF 15430]

No criterion for the legitimacy of certain probability computations has been furnished by the classical theory of probability; hence the multitude of paradoxes. Such criteria were first introduced in the frequency or sequential theories of probability. In the present paper the author shows how his (frequency) theory can be applied to decide the legitimacy of the classical computation for the probability of winning a set of games. The paper was written partly to point out the need for such criteria and partly to answer a claim by Greville [Trans. Amer. Math. Soc. 46, 410–425 (1939); these Rev. 1, 61] that the author's theory was not sufficiently general to include such a criterion.
A. H. Copeland (Ann Arbor, Mich.).

Cox, R. T. Probability, frequency and reasonable expectation. Amer. J. Phys. 14, 1–13 (1946). [MF 15813]

***Fréchet, Maurice. Les probabilités associées à un système d'événements compatibles et dépendants. II. Cas particuliers et applications.** Actualités Sci. Ind., no. 942. Hermann et Cie., Paris, 1943. 131 pp. [paged 81–211].

This is the second of a projected series of three parts. Part I was subtitled "Événements en nombre fini fixe" [Actualités Sci. Ind., no. 859; these Rev. 3, 168] and part III is to be "Événements en nombre très grande ou infini." The author has given substantial extracts from part II in Revista Mat. Hisp.-Amer. (4) 4, 95–126 (1944) [these Rev. 6, 231].

In the introduction the author gives a summary of the notation and formulas of part I and, in a separate preface, some corrections and extensions. Chapter III (the numbering follows that of part I) deals with the binomial distribution and, more generally, sets of events for which the joint probability of r events is a function of r alone. The author derives various inequalities, and formulas connected with runs. Chapter IV is devoted to several special combinatorial problems.

The fifth and most extensive chapter deals with the "problème des rencontres," that is, the probability of a specified number of coincidences in matching two decks of cards. Basing himself on a symbolic method due to the

reviewer [Amer. Math. Monthly **46**, 159–161 (1939)], the author develops the theory at considerable length. An independent treatment which overlaps to some extent has been given by the reviewer [Bull. Amer. Math. Soc. **50**, 906–914 (1944); these Rev. **6**, 159]. No reference is made to the extensive literature which has grown up around the Duke telepathy experiments [for a bibliography cf. Battin, Ann. Math. Statistics **13**, 294–305 (1942); these Rev. **4**, 102]; many of these papers were probably not available to the author.

Two brief appendices are devoted to the problem of moments and the Laguerre polynomials.

Table of contents by chapters: III. Cas particuliers: schéma de Bernoulli, problème de Bernoulli pour plusieurs événements, événements échangeables. IV. Premières applications. V. Différents jeux de coincidences: problème classique des rencontres, premières généralizations classiques du jeu des rencontres, formes plus générales du jeu des rencontres, factorization interne des probabilités associées au jeu des coincidences, jeu des coincidences multiples, une généralization analytique du jeu de rencontre.

I. Kaplansky (Princeton, N. J.).

Fan, Ky. Conditions d'existence de suites illimitées d'événements correspondant à certaines probabilités données. Revue Sci. (Rev. Rose Illus.) **82**, 235–240 (1944). [MF 16273]

Necessary and sufficient conditions are found on a set of numbers $P_{[r]}^{(m)}$ so that $P_{[r]}^{(m)}$ can be interpreted as the probability that exactly r of the first m of a sequence of random events occur; necessary and sufficient conditions are also found when "exactly" is replaced by "at least"; two related sets of numbers are also treated completely. [For earlier work see the book reviewed above.] Particular cases are then treated: where the events are equivalent, that is, if the probability of the occurrence of any r events depends only on r; and a slightly more general case. *J. L. Doob.*

Baticle, Edgar. Le problème des stocks. C. R. Acad. Sci. Paris **222**, 355–357 (1946). [MF 16009]

The problem in question is to determine the number of ways of putting m objects into n boxes so that p boxes get at least q. The author's three previous notes [same C. R. **196**, 1945–1946; **197**, 632–634 (1933); **201**, 862–864 (1935)] are amplified by giving asymptotic expressions and formulas for mean and variance. *I. Kaplansky* (Princeton, N. J.).

Olmstead, P. S. Distribution of sample arrangements for runs up and down. Ann. Math. Statistics **17**, 24–33 (1946). [MF 15955]

Consider the $n!$ permutations of n different numbers and observe the signs of the differences of consecutive numbers. A sequence of p consecutive identical signs preceded and followed by different signs is called a run up or down (as the case may be) of length p. The author is interested in the number of permutations which contain a run of length at least p. He gives, among other things, a table for $p<n\leqq 14$, a discussion for large n and $(p+1)!/n$ constant and an extrapolation with very good accuracy to small n. The results are of value in quality control. *J. Wolfowitz.*

Treloar, L. R. G. The statistical length of long-chain molecules. Trans. Faraday Soc. **42**, 77–82 (1946). [MF 16554]

Let a chain in 3-space consist of n links of the same length l and of random orientation, all directions being equally likely. Let $P_n(r)$ be the probability density that the two endpoints are at a distance r, and let $p_n(x)$ be the probability density that the distance of the projections of the endpoints onto a fixed direction is x. It is shown that $p_n(x)$ is the n-fold convolution of the rectangular distribution with itself, and $P_n(x)=-2xp_n'(x)$ for $x>0$. Hence explicit formulas can be obtained for both functions. Numerical values are given for $n=25$ and $n=100$. *W. Feller.*

Rott, N. Ueber Wahrscheinlichkeitsprobleme der Garnfestigkeitsprüfung. Schweiz. Arch. Angew. Wiss. Tech. **12**, 93–95 (1946). [MF 16139]

It is shown that under rather general conditions the number of flaws in a thread will satisfy a Poisson distribution. The theory is compared with observations.

W. Feller (Ithaca, N. Y.).

Schrödinger, Erwin. Probability problems in nuclear chemistry. Proc. Roy. Irish Acad. Sect. A. **51**, 1–8 (1945). [MF 16686]

A nucleus is split by the chance hit of an erratic particle and gives birth to a number N of particles of the same kind. Each of these has probability $1-\zeta$ of being removed without a new efficient collision and probability ζ of repeating the same process. Assuming statistical independence for all particles the author proves by elementary considerations that the probability p that the process will stop dead after finitely many generations is one in case $N\zeta<1$ and equals the least positive root of $x=(\zeta x+1-\zeta)^N$ if $N\zeta>1$. This simple fact is generalized to the case where the probabilities vary in space so that there is a probability density $f(\mathbf{r}, \mathbf{r}')$ that a particle produced at $\mathbf{r}$ scores a splitting hit at $\mathbf{r}'$. The relevant analogue to ζ seems then determined by the first characteristic function of an integral equation. The theory can be made rigorous in the case of a symmetric function $f(\mathbf{r}, \mathbf{r}')$, and at least plausible in general. *W. Feller.*

Cernuschi, Felix, and Castagnetto, Louis. Chains of rare events. Ann. Math. Statistics **17**, 53–61 (1946). [MF 15958]

The authors study random events which (like automobile accidents) can be classified as simple or multiple according to the number of units involved. It is supposed that the numbers of s-tuple events ($s=1, 2, \cdots$) represent mutually independent random variables each having a Poisson distribution with parameter λ_s. The resulting distribution $\{P(n)\}$ of the total number of units is studied with particular reference to the special case $\lambda_s=\lambda_1 a^{s-1}/s!$. Using the saddle point method an approximation to $P(n)$ for large n is obtained. *W. Feller* (Ithaca, N. Y.).

Neyman, Jerzy. Un théorème d'existence. C. R. Acad. Sci. Paris **222**, 843–845 (1946). [MF 16278]

Let $X_1, \cdots, X_n$ be mutually independent chance variables with continuous distribution functions. It is shown that there is a measurable function $f(t)$ such that $f(X_j)$ is uniformly distributed between 0 and 1 for every j.

J. L. Doob (Urbana, Ill.).

Iegudin, G. Parameters of distribution of a random variable invariant under translations, and algebraic semi-invariants. C. R. (Doklady) Acad. Sci. URSS (N.S.) **48**, 615–617 (1945). [MF 16639]

Let X be a random variable with moments $m_1, m_2, \cdots$ and consider the binary form $F=\sum_{h=1}^{n}\binom{n}{h}m_h x^{n-h}y^h$. Continuing the investigations of P. L. Dressel [Ann. Math.

Statistics 11, 33–57 (1940); these Rev. 1, 249] the author proves that a function $\theta_r = f(m_1, \cdots, m_r)$ with $r \leqq n$ rests invariant under translations of X if and only if it is a semi-invariant of F, that is to say, rests invariant under transformations $x = \xi + a\eta$, $y = \eta$. Applications to unbiased estimates of moment functions are given. *W. Feller.*

Loève, Michel. Étude asymptotique des sommes de variables aléatoires liées. J. Math. Pures Appl. (9) 24, 249–318 (1945). [MF 15973]

This paper (written in 1941) contains a systematic and original study of the classical limit theorems for the case of dependent random variables. It contains many results of various degrees of generality and criteria of a new form. It is only possible to review the main results briefly here. The first chapter is devoted to sequences of events A_k, that is to say, to random variables X_k which assume the values zero or one according as A_k is not or is realized. Put

$$\sum_{k=1}^{n} \Pr(A_k) = np_1(n), \qquad \sum_{1 \leqq i < k \leqq n} \Pr(A_i A_k) = \binom{n}{2} p_2(n)$$

and $d_n^2 = p_2(n) - p_1^2(n)$. It is shown that a necessary and sufficient condition for the stability of the means

$$\{X_1 + \cdots + X_n\}/n$$

is that $d_n \to 0$; if $nd_n^2 = O(1)$, the stability is strong. Suppose next that for some r (fixed or varying with n)

$$\varliminf_{\nu\to\infty} \varliminf_{n\to\infty} \frac{(r-1)!(n-r)!}{(n-1)!} \sum \Pr(A_{\nu+i_1} + \cdots + A_{\nu+i_r}) \leqq p,$$

where the summation extends over $1 \leqq i_1 < i_2 < \cdots < i_r \leqq n$; then the probability of the realization of infinitely many A_k is at most p. A similar statement is true with the inequalities reversed and the inferior limits replaced by superior limits.

The following chapters deal with the familiar double sequences of random variables $X_{n,k}$ with $1 \leqq k \leqq n$, $n = 1, 2, \cdots$. Put $S_{n,k} = \sum_{\nu=1}^{k} X_{n,\nu}$. It is assumed that the $X_{n,k}$ have vanishing expectations. The conditional expectation of $X_{n,k}$ knowing $S_{n,k-1}$ is denoted by $E'(X_{n,k})$. The sequence $X_{n,k}$ is called asymptotically independent of first order or AI_1 if

$$(*) \qquad \sum_{k=1}^{n} \sup |E'(X_{n,k})| \to 0.$$

The sequence is called asymptotically independent in the mean if

$$\lim_{n\to\infty} \sum_{1 \leqq i < k \leqq n} E(X_{n,i} X_{n,k}) = 0.$$

In the second chapter various criteria for both the weak and the strong law of large numbers are given for such sequences. In particular, Kolmogoroff's condition for the strong law is generalized to AI_1 sequences. The third chapter is devoted to the central limit theorem. Sufficient conditions are given in a form analogous to Feller's necessary and sufficient conditions for the case of independent variables. The only change is that the distribution function $F_{n,k}(x)$ of $X_{n,k}$ is replaced by the conditional distribution function $F_{n,k}(x)$ knowing $S_{n,k-1}$ and all integrals are replaced by their upper bounds exactly as in (*). The proof is independent of previous results and even new for the case of independent variables. Several alternative forms of the criterion are given.

The fourth chapter opens with various inequalities for moments of sums of dependent variables. They are used to deduce several new sets of sufficient conditions for the central limit theorem, all of which generalize the classical results of S. Bernstein. Following Bernstein, nothing is assumed concerning the mutual dependence of $X_{n,i}$ and $X_{n,k}$ for $|i-k| < d_n$; however, a gradual weakening of this stochastic dependence is postulated for $|i-k| > d_n$. Various ways of rendering the last vague statement precise lead to various new types of sufficient conditions. *W. Feller.*

Loève, Michel. Remarques sur les ensembles de lois. C. R. Acad. Sci. Paris 222, 628–630 (1946). [MF 16041]

The author considers chance variables which may take on a single infinite value. He finds conditions that distribution functions F_n for these variables should converge to a distribution function F at the points of continuity of F, with and without the condition that the probability of taking on infinity should converge to the corresponding probability. For example (without this condition), there is convergence if and only if, for each t,

$$\int_{-\infty}^{\xi} e^{iux} dF_n(x) \to \int_{-\infty}^{\xi} e^{iux} dF(x)$$

for all large ξ which are points of continuity of F. Finally, using methods of the first of the two papers reviewed below, he derives a simple proof, imposing fewer restrictions on the functions involved, of a theorem of Boas and Kac [Duke Math. J. 12, 189–206 (1945), theorem 6; these Rev. 6, 265] that, if $f(x)$ is the cosine transform of a nonnegative function, then, for any real c and β, $f(0) + 2\sum_1^\infty f(n\beta) \cos 2\pi nc \geqq 0$. *J. L. Doob.*

Loève, Michel. Quelques propriétés des fonctions aléatoires de second ordre. C. R. Acad. Sci. Paris 222, 469–470 (1946). [MF 16024]

Loève, Michel. Sur les fonctions aléatoires vectorielles de second ordre. C. R. Acad. Sci. Paris 222, 942–944 (1946). [MF 16318]

The author continues his studies on stochastic processes depending on a continuous parameter, that is, on continuous parameter families of chance variables $X(\alpha)$ [cf. the same C. R. 220, 295–296, 380–382 (1945); these Rev. 7, 129]. Proofs are omitted. Decomposition theorems are outlined which express processes as linear combinations of uniquely determined orthogonal processes; sufficient conditions are found for the strong law of large numbers, in terms of the correlation function of the process. The correlation matrix of the successive derivatives $X'(\alpha), \cdots$ is used to obtain a simple difference differential equation in orthogonalized linear combinations of the successive derivatives. Finally, if for each α the chance variable $X(\alpha)$ takes on only denumerably many values $X_1(\alpha), X_2(\alpha), \cdots$ with probabilities $p_1(\alpha), p_2(\alpha), \cdots$ the chance variable is thought of (for each α) as a point in a Cartesian space with coordinates $(x_1, x_2, \cdots)$ whose coordinates are given weights $p_1(\alpha)$, $p_2(\alpha), \cdots$. The process then corresponds to a curve and generalized Frenet formulas for this curve are derived. The generalization to N-dimensional chance vectors is indicated. *J. L. Doob* (Urbana, Ill.).

Bergström, Harald. On the central limit theorem. Skand. Aktuarietidskr. 27, 139–153 (1944). [MF 14147]

An alternative derivation of the Berry-Esseen estimate of the difference between the distribution function of a sum of n independent random variables and the normal distribution. [Cf. A. C. Berry, Trans. Amer. Math. Soc. 49, 122–136 (1941); C.-G. Esseen, Ark. Mat. Astr. Fys. 28A, no. 9 (1942); these Rev. 2, 228; 6, 232. See also P. L. Hsu, Ann. Math. Statistics 16, 1–29 (1945); these Rev. 6, 233.] *M. Kac* (Ithaca, N. Y.).

Bergström, Harald. On the central limit theorem in the space R_k, $k>1$. Skand. Aktuarietidskr. **28**, 106–127 (1945). [MF 14156]

Extension of the methods of the paper reviewed above to the multidimensional case. *M. Kac* (Ithaca, N. Y.).

Erdös, P., and Kac, M. On certain limit theorems of the theory of probability. Bull. Amer. Math. Soc. **52**, 292–302 (1946). [MF 16193]

Soit la suite $s_k=X_1+X_2+\cdots+X_k$, $k=1, 2, \cdots$, de sommes des variables aléatoires indépendantes et équiprobables (ou seulement telles que le théorème limite central s'applique), de valeur moyenne nulle et d'écart-type unité. On étudie, lorsque $n\to\infty$, les lois limites des expressions suivantes:

$$\max n^{-\frac{1}{2}}(s_1, s_2, \cdots, s_n), \tag{1}$$
$$\max n^{-\frac{1}{2}}(|s_1|, |s_2|, \cdots, |s_n|), \tag{2}$$
$$n^{-2}(s_1^2+s_2^2+\cdots+s_n^2), \tag{3}$$
$$n^{-\frac{3}{2}}(|s_1|+|s_2|+\cdots+|s_n|). \tag{4}$$

Les auteurs démontrent que, grâce à la tendance centrale, ces lois existent et sont indépendantes de celle des X_k. En choisissant convenablement cette dernière ils calculent alors les fonctions de distribution des trois premières et la transformée réelle de Laplace de la fonction de distribution de la quatrième. On ramène ainsi (1) à la ruine des joueurs, (2) à la promenade au hasard entre deux barrières absorbantes, (4) à un problème étudié ailleurs par l'un des auteurs. Pour (3) on utilise la méthode des fonctions caractéristiques.

Les auteurs signalent que leurs resultats généralisent ceux de Bachelier (pour (1) and (2)) et de Cameron et Martin (pour (3)). Il y a lieu de rapprocher le théoréme I d'un lemme de P. Lévy et Cantelli relatif à la loi du logarithme itéré [voir, par example, Cantelli, Giorn. Ist. Ital. Attuari **4**, 327–350 (1933), particulièrement, pp. 340–343].

M. Loève (London).

Feller, W. On the normal approximation to the binomial distribution. Ann. Math. Statistics **16**, 319–329 (1945). [MF 15465]

Reprenant le problème de l'approximation d'une loi binomiale par une loi normale, l'auteur envisage le problème suivant: soient $T_k=\binom{n}{k}p^kq^{n-k}$ $(0<p<1,\ q=1-p)$, $P_{\lambda\nu}=\sum_{k=\lambda}^{\nu}T_k$ $(\lambda<\nu)$, déterminer deux nombres α et β, fonctions simples de λ et ν, et tels que l'erreur commise en remplaçant $P_{\lambda\nu}$ par $\phi(\beta)-\phi(\alpha)$, où

$$\phi(x)=(2\pi)^{-\frac{1}{2}}\int_{-\infty}^{x}e^{-u^2/2}du,$$

soit minima. À ce point de vue, il obtient le résultat suivant: posons $\sigma^2=(n+1)pq$, $a=(p-q)/6$; si $\sigma>3$ et si $\lambda\geqq(n+1)p$, $\nu+\frac{1}{2}\leqq(n+1)p+\frac{2}{3}\sigma^2$, on a: (a) $P_{\lambda\nu}\leqq e^{5(1-pq)/36\sigma^2}\{\phi(\beta)-\phi(\alpha)\}$ en prenant $\alpha=\eta_\lambda$, $\beta=\eta_{\nu+1}$, où η_k désigne l'expression

$$\frac{k-(n+1)p}{\sigma}+\frac{a}{\sigma}\left\{\frac{k-(n+1)p}{\sigma}\right\}^2+\frac{2a}{\sigma}-\frac{1}{2\sigma^2};$$

(b) $P_{\lambda\nu}\geqq e^{5(1-pq)/36\sigma^2}\{\phi(\beta)-\phi(\alpha)\}$ en prenant $\alpha=\eta_\lambda$, $\beta=\eta_{\nu+1}$, où η_k désigne l'expression

$$\frac{k-(n+1)p}{\sigma}+\frac{a}{\sigma}\left\{\frac{k-(n+1)p}{\sigma}\right\}^2+\frac{2a}{\sigma}+\frac{M}{6\sigma}+\frac{1}{7\sigma},$$

où $M=\{\nu+\frac{1}{2}-(n+1)p\}^3/\sigma^4$. La démonstration repose sur une évaluation approchée nouvelle de T_k; les principaux avantages de ce théorème, par rapport à ceux déjà connus, sont d'être applicable pour de petites valeurs de σ et de fournir une évaluation de l'erreur relative. *R. Fortet.*

Lévy, Paul. Les processus fortement continus et la loi de Laplace. C. R. Acad. Sci. Paris **222**, 839–841 (1946). [MF 16276]

Let $Y(t)$ be a chance variable for each t in some interval and suppose that the $Y(t)$ process has independent stationary increments, in the sense that $Y(t+h)-Y(t)$ is independent of previous increments and has a distribution not depending on t. The concepts of infinitesimal $\delta Y(t)$ and of tangency of two processes at a given t value are introduced in such a way that if there is a $Y(t)$ process whose sample functions are almost all continuous, and if it is tangent at t_0 to an $X(t)$ process with independent stationary increments, then $\delta X(t_0)$ is normally distributed. This is a generalization of the fact, which follows from the central limit theorem, that, if almost all the sample functions of the $X(t)$ process are continuous, the increments $X(t+h)-X(t)$ are normally distributed. *J. L. Doob* (Urbana, Ill.).

Savkevitch, V. Schèma de l'urne aux boules surajoutées. Leningrad State Univ. Annals [Uchenye Zapiski] **83** [Math. Ser. 12], 129–149 (1941). (Russian. French summary) [MF 16493]

Details of the proofs of the results stated in C. R. (Doklady) Acad. Sci. URSS (N.S.) **28**, 8–12 (1940) [these Rev. **2**, 229]. *W. Feller* (Ithaca, N. Y.).

Onicescu, O., et Mihoc, G. Le coefficient de dispersion et la dépendance des épreuves. Bull. Math. Soc. Roumaine Sci. **46**, 77–80 (1944). [MF 16510]

The authors exhibit an example of a Markov chain of the second order (dependence of result of a trial on those of two previous trials) for which there is normal dispersion, that is, Lexis coefficient 1. This shows that normality of dispersion is insufficient to characterize Bernoulli trials.

J. L. Doob (Urbana, Ill.).

Sarymsakov, T. A. Sur une synthèse des deux méthodes d'exposer la théorie des chaînes discrètes de Markoff. C. R. (Doklady) Acad. Sci. URSS (N.S.) **48**, 159–161 (1945). [MF 16663]

Exposition of an appropriate order of the fundamental theorems of finite Markov chains in which both the qualitative and the quantitative aspects become clear.

W. Feller (Ithaca, N. Y.).

Sarymsakov, T. Un nouveau critère nécessaire et suffisant pour la régularité des chaînes de Markoff dont l'ensemble des états possibles est continu. C. R. (Doklady) Acad. Sci. URSS (N.S.) **49**, 85–88 (1945). [MF 16401]

Consider the integral equation

$$\varphi(x)=\lambda\int_a^b K(x, s)\varphi(s)d\eta(s),$$

where $\eta(s)$ is increasing and the kernel is nonnegative and continuous in $a\leqq x, s\leqq b$. In order that there exists a positive characteristic function $\varphi_0(x)$ belonging to the characteristic value λ_0 which is smallest in absolute value and that this λ_0 is positive and a simple root of the Fredholm equation, it is necessary and sufficient that for all m sufficiently large the iterated kernels satisfy the relation $K^{(m)}(x, x)>0$. Using this theorem and a criterion due to M. Fréchet [Bull. Soc. Math. France **62**, 68–83 (1934)] the author concludes that a Markov chain with transition probabilities $p(x, y)$ is

regularly ergodic if, and only if, $p^{(m)}(x, x)>0$ for all m sufficiently large. *W. Feller* (Ithaca, N. Y.).

Blanc-Lapierre, André, et Fortet, Robert. Sur la décomposition spectrale des fonctions aléatoires stationnaires d'ordre deux. C. R. Acad. Sci. Paris **222**, 467–468 (1946). [MF 16023]

Let $\{x(t)\}$, $-\infty<t<+\infty$, be the chance variables of a stationary stochastic process [Khintchine, Math. Ann. **109**, 604–615 (1934)]. A linear filter, defined by a function $R(t)$ transforming $x(t)$ into $x_1(t)$ in accordance with

$$x_1(t)=\int_{-\infty}^{\infty} x(\theta)R(t-\theta)d\theta,$$

is characterized by its gain $G(\omega)$, the Fourier transform of R. The authors discuss the basic properties of filters. The frequency spectrum intensities of $x(t)$ are multiplied by $|G|^2$; to a succession of filters corresponds the product of their gain functions, etc. The properties described have appeared in the literature in various particular cases. The identity between the frequency analysis of $x(t)$ and the spectral analysis of groups of unitary transformations of Hilbert space $\{T_s\}$ with $T_{s+t}=T_sT_t$ is noted. *J. L. Doob.*

Blanc-Lapierre, André, et Fortet, Robert. Résultats sur la décomposition spectrale des fonctions aléatoires stationnaires d'ordre 2. C. R. Acad. Sci. Paris **222**, 713–714 (1946). [MF 16172]

Further development of the ideas quoted in the preceding review. The gain functions corresponding to simple operations on $x(t)$ (such as derivation) are noted. If the frequency spectrum of the process is confined to a small interval, $x(t)$ is locally essentially a sine wave whose amplitude is easily calculated. Any process can be decomposed into processes of this type. Under certain conditions almost all sample functions have frequency spectra identical with that of the process. Details and proofs are omitted. [Cf. the related work of S. O. Rice, Bell System Tech. J. **23**, 282–332 (1944); **24**, 46–156 (1945); these Rev. **6**, 89, 233, and, in connection with the last point, Wiener and Wintner, Amer. J. Math. **63**, 794–824 (1941); these Rev. **4**, 15.] *J. L. Doob.*

Blanc-Lapierre, André, et Fortet, Robert. Extension de la méthode des filtres à des fonctions aléatoires non stationnaires. C. R. Acad. Sci. Paris **222**, 1270–1271 (1946). [MF 16731]

The hypothesis that the process is stationary is found to be unnecessary for many of the results of the two papers reviewed above. In fact, if the expectation of $x(t)$ $(E\{x(t)\})$ vanishes identically in t, if

$$\lim_{T\to\infty} T^{-1}\int_h^{T+h} E\{x(t)x(t+\tau)\}dt=\rho(\tau)$$

uniformly in h and τ, and if certain hypotheses implying the rapid attenuation of the influence of the past on the future are imposed, then $\rho(t)$ is the Fourier-Stieltjes transform of a monotone function which as usual describes the frequency spectrum of the process; $\lim_{T\to\infty} T^{-1}\int_0^T x(t)dt=0$ and $\lim_{T\to\infty} T^{-1}\int_0^T x(t)x(t+\tau)dt=\rho(\tau)$ with probability 1. The above hypotheses are so phrased that, if $x(t)$ is transformed into $x_1(t)$ by a linear filter defined by an absolutely integrable function $R(t)$ [cf. the first of the preceding two reviews], then the hypotheses will also be true of the $x_1(t)$ process. *J. L. Doob* (Urbana, Ill.).

Blanc-Lapierre, André, et Lapostolle, Pierre. Fluctuations dans les grandeurs physiques quasi sinusoïdales. C. R. Acad. Sci. Paris **222**, 1324–1325 (1946). [MF 16736]

The deflection of a pendulum (with small damping) excited by impulses $\{q_j\}$ at times $\{t_j\}$ is considered. (a) The impulses are supposed mutually independent chance variables with a common distribution function, and the t_j are supposed very close to the times of passing through the equilibrium position with positive velocity. (b) The q_j are supposed constant and all equal; the t_j fluctuate about equally spaced points or are distributed in accordance with the Poisson distribution. In both cases the frequency spectrum and associated characteristics are evaluated. Finally, the electromotive force $e(t)$ of an alternator whose velocity varies at random in a stationary process is considered. The correlation function of the $e(t)$ process is found and its frequency spectrum evaluated under special conditions. *J. L. Doob* (Urbana, Ill.).

Bass, Jean. Les fonctions aléatoires et leur interprétation mécanique. Revue Sci. **83**, 3–20 (1945). [MF 16374]

The author reviews the recent progress made by Dedebant, Wehrlé and Moyal in the mechanical interpretation of certain stochastic processes and conversely in the stochastic interpretation of certain physical phenomena. Suppose that, for each t, $x(t)$ is a chance variable, with density of distribution $\rho(x, t)$. Suppose that $\bar{u}(x, t)$ is the conditional expectation of the velocity $x'(t)$ for given $x(t)=x$. Then there is an equation of continuity (1) $\partial\rho/\partial t+\partial(\rho\bar{u})/\partial x=0$. If it is supposed that $x''(t)$ exists, if $R(x, u, t)$ is the joint probability density of $x(t)$ and $x'(t)$, and if $A(x, u, t)$ is the conditional expectation of $x''(t)$ for given $x(t)=x$, $x'(t)=u$, then there is a further equation, called the equation of structure, (2) $\partial R/\partial t+u\partial R/\partial x+\partial(AR)/\partial u=0$. There is a strong analogy between a stochastic process with $x'(t)$ existing and the random motion of a fluid. In fact, for such a process (in 3 dimensions) there is a conditional distribution of velocity at each point, and thus a random velocity field in space, that is, a process whose sample function is a velocity field. This suggestive analogy can be carried through in considerable detail. For example, (2) leads to a general equation of transfer, the analogue of Boltzmann's equation in the kinetic theory of gases, for any function of position and velocity. When $\psi=1$, this becomes (1). When ψ is a velocity component, this becomes the corresponding Navier-Stokes equation of hydrodynamics. When ψ is suitably chosen, this becomes the analogue of the Clausius equation in thermodynamics for the rate of change of entropy. A transfer equation can always be found even when (2) is replaced by (2*) in which there is an operator $D(R)$ on the right, by multiplying the structure equation by ψ and taking the conditional expectation for fixed positions of the moving point. In this way a general equation is always obtained which has simple physical interpretations for suitable choices of ψ. The idea is applied to processes whose velocity processes are Markov processes, so that the equation of structure is the Kolmogoroff equation for the transition probabilities, and to a general class of forms for $D(R)$. The author considers that (2) is true under very general hypotheses, with proper interpretation of A, whenever $x'(t)$ exists. In this way a wide class of physical phenomena can be represented by the random motion of a particle. The operator $D(R)$ serves to characterize the problems of "stochastic mechanics" thus derived. *J. L. Doob* (Urbana, Ill.).

Frenkiel, François N. Étude statistique de la turbulence; théorie de la mesure de la turbulence avec un seul fil chaud non compensé. C. R. Acad. Sci. Paris **222**, 585–587 (1946). [MF 16037]

Without compensation a hot wire anemometer reduces the kinetic energy of turbulent velocity fluctuations of frequency ω to an amount $1/(1+M^2\omega^2)$. The longitudinal spectrum of turbulence being $f(\omega)$, the amount E_c of the total kinetic energy E given by the hot wire is

$$E_c/E=\int_0^\infty \frac{f(s)}{1+M^2s^2}ds.$$

The reviewer [in a paper still unpublished] has established that, introducing the correlation $R(h)$, this formula is equivalent to:

$$E_c/E=M^{-1}\int_0^\infty \exp(-s/M)R(s)ds.$$

The author gives several applications of that result.

J. Kampé de Fériet (Lille).

Frenkiel, François N. Étude statistique de la turbulence: corrélation et spectres dans un écoulement homogène. C. R. Acad. Sci. Paris **222**, 367–369 (1946). [MF 16014]

Frenkiel, François N. Études statistiques de la turbulence: corrélations et spectres dans un écoulement de turbulence homogène et isotrope. C. R. Acad. Sci. Paris **222**, 473–475 (1946). [MF 16026]

The author recalls the various definitions of the correlation and the spectrum of turbulence; discussing the longitudinal and transversal correlations R_x and R_y of von Kármán [J. Aeronaut. Sci. **4**, 131–138 (1937)] and the corresponding scales of turbulence L_x and L_y, he proves that $L_x=2L_y$; he also gives relations between the longitudinal and transversal spectra $f_x(\omega)$ and $f_y(\omega)$ related to R_x and R_y.

J. Kampé de Fériet (Lille).

Vaulot, Émile. Délais d'attente des appels téléphoniques traités au hasard. C. R. Acad. Sci. Paris **222**, 268–269 (1946). [MF 15998]

Pollaczek, Félix. La loi d'attente des appels téléphoniques. C. R. Acad. Sci. Paris **222**, 353–355 (1946). [MF 16008]

Let $k\geqq 1$ telephone trunk lines serve customers and suppose that the incoming calls have a Poisson distribution with a parameter $\eta<k$. It is furthermore assumed that the probability that the duration of a conversation exceeds t is e^{-t}; it is well known that with these "exponential holding times" the process becomes a Markov process. If all k lines happen to be busy, each new call is directed to stand by in a waiting line. When one of the k trunk lines becomes free, each individual in the waiting line has the same probability of being served. Let $f_n(t)$ be the probability that an individual who at time 0 is waiting along with n more individuals will not yet be served at time t ($n=0, 1, 2, \cdots$). In the first paper the differential equations

$$f_n'=-(k+\eta)f_n+\frac{kn}{n+1}f_{n-1}+\eta f_{n+1}$$

are derived; the initial conditions are $f_n(0)=1$ and $f_{-1}(t)=0$. In the second paper it is shown that the generating function $\sum f_n(t)x^n$ satisfies a certain differential equation and that its Laplace transform can be represented in the form of an integral. In this way at least an asymptotic expansion of the generating function for $t\to\infty$ can be obtained explicitly.

W. Feller (Ithaca, N. Y.).

Reboul, Georges, et Reboul, Jean-Antoine. Application des relations de probabilités aux équilibres physiques et biologiques. C. R. Acad. Sci. Paris **222**, 1063–1066 (1946). [MF 16521]

Reboul, Georges. Relations de probabilités dans les cas d'interdépendance. Applications à la chimie. C. R. Acad. Sci. Paris **222**, 1320–1322 (1946). [MF 16735]

Continuing their investigations [J. Phys. Radium (8) **5**, 108–116 (1944); these Rev. **6**, 231] the authors find that "un changement fini de l'état d'un système physique ou biologique est toujours la somme intégrale de changements élémentaires uniquement régis par des lois de hasard." This leads to a generalized Gibbs law that the number of independent variables is $n+p-\phi$, where n, p, and ϕ are the numbers of constituents, factors of action, and phases, respectively.

W. Feller (Ithaca, N. Y.).

Mathematical Statistics

Wiener, Norbert. The theory of statistical extrapolation. Bol. Soc. Mat. Mexicana **2**, 37–42 (1945). (Spanish) [MF 14478]

The author considers the problem of extrapolating a sequence P of numbers P_k ($-\infty<k\leqq 0$) to positive values of k. A measure invariant under the transformation $T:P_k\to P_{k+1}$ is defined in the space of sequences $P=\{P_k\}$, $-\infty<k<\infty$, and it is assumed that the measure is ergodic, so that the equation

$$(1)\qquad \int f(P)dV_p=\lim_{n\to\infty}\{f(Q)+f(TQ)+\cdots+f(T^nQ)\}/(n+1)$$

holds for all Q except a set of measure zero. By means of (1), the mean value and probability distribution for a function $f(P)$ with fixed past sequence P_k ($k\leqq 0$) can be expressed by means of linear operations on the functions which depend only on the past. This leads to a process, realizable at least in theory by an electrical or mechanical circuit, for computing the mean and distribution of a future value P_k ($k\geqq 1$) when the past is known.

W. Kaplan (Ann Arbor, Mich.).

Hsu, L. C. Some combinatorial formulas on mathematical expectation. Ann. Math. Statistics **16**, 369–380 (1945). [MF 15470]

This is the same as the author's paper in the same Ann. **15**, 399–413 (1944) [these Rev. **6**, 234].

Z. W. Birnbaum (Seattle, Wash.).

Kreis, H. Beitrag zur Theorie der Häufigkeitsfunktionen. Mitt. Verein. Schweiz. Versich.-Math. **45**, 239–256 (1945). [MF 15420]

This paper contains an elaboration of one of the familiar symbolic methods for computing moments of a binomial distribution. It also contains a discussion of discrete and continuous distributions.

A. H. Copeland.

Hald, A. Hjorth. The truncated normal distribution. Mat. Tidsskr. B. **1946**, 83–91 (1946). (Danish) [MF 16309]

Best estimates for the parameters of the truncated normal distribution are described and the second moments of these statistics are computed.

W. Feller (Ithaca, N. Y.).

Jordan, Charles. Remarques sur la loi des erreurs. Acta Univ. Szeged. Sect. Sci. Math. 10, 112–133 (1941). [MF 15839]

Remarks on the significance of the normal distribution in statistics. The variance ξ and mean z are given a priori distributions with densities $1/\xi^n$ and 1, respectively.

J. L. Doob (Urbana, Ill.).

Darmois, Georges. Sur certaines lois de probabilité. C. R. Acad. Sci. Paris 222, 164–165 (1946). [MF 15987]

The author discusses the properties of an (x, y) distribution depending on a parameter m implied by the existence of a continuous valued chance variable whose distribution is independent of m. If x itself is such a chance variable, the density of distribution of (x, y) can be put in the form $A(x)B_x(y, m)$, where A is the density of distribution of x and B that of y for given x. This is compared with the situation when x is a sufficient statistic for the estimation of m; in that case A depends on m but B does not.

J. L. Doob (Urbana, Ill.).

Darmois, Georges. Résumés exhaustifs et problème du Nil. C. R. Acad. Sci. Paris 222, 266–268 (1946). [MF 15997]

Let (x_i, y_i), $i=1, \cdots, n$, be a sample of n pairs from a bivariate distribution depending on a parameter m. Then it may happen that there is a pair of chance variables X, Y, functions of the sample values, which are sufficient for the estimation of m, that is, the conditional distribution of any other chance variable for preassigned X, Y does not depend on m. Examples are given (a) of (x, y) distributions as in the preceding review for which there are also sufficient pairs of statistics and (b) of (x, y) distributions as in (a) for which in addition the sufficient pairs X, Y themselves have the property described in the preceding review.

J. L. Doob (Urbana, Ill.).

Féraud, Lucien. Sur les distributions à projection indépendante du paramètre. C. R. Acad. Sci. Paris 222, 1272–1273 (1946). [MF 16732]

The author gives a sufficient condition that a bivariate density distribution $f(x, y, m)$ depending on the parameter m can be reduced by a change of variables $(x, y)\rightarrow(u, v)$ to a density of the form $F(u, v-m)$. [Cf. the two preceding reviews for a discussion of the significance of bivariate distributions depending on a parameter, for which there is a variate u whose distribution is independent of the parameter.]

J. L. Doob (Urbana, Ill.).

Ghosh, Birendranath. On the construction of some natural fields. Science and Culture 9, 213–214 (1943).

Knobloch, H. Funktionsgewichte in der Ausgleichsrechnung. Z. Angew. Math. Mech. 21, 315–316 (1941). [MF 15861]

The author assumes that any $n-1$ of the variables $a_1, \cdots, a_n$ determine x, $x=f_j(a_1, \cdots, a_{j-1}, a_{j+1}, \cdots, a_n)$ and attacks the problem of finding suitable weights for linear combinations of the f_j to approximate x, if $a_1, \cdots, a_n$ are measured and the measured values substituted in the f_j. He takes the weights proportional to the reciprocals of the dispersions of the f_j, and, assuming small dispersions of the a_j (an unstated hypothesis), he then can find the weights in terms of the dispersions of the a_j and the partial derivatives of the f_j. A simple method of calculating these partial derivatives is given. It is not clear how to justify the author's choice of weights without an analysis of the stochastic independence of the f_j.

J. L. Doob (Urbana, Ill.).

Berkson, Joseph. Approximation of chi-square by "probits" and by "logits." J. Amer. Statist. Assoc. 41, 70–74 (1946). [MF 16298]

The author [same J. 39, 357–365 (1944)] introduced a method of fitting the logistic curve to observed rates (for example, dosage-mortality data) by logits, which is analogous to the probit method of fitting a cumulative normal curve to such data. If the ultimate aim of fitting is taken as the minimization of χ^2, calculated for observed rates versus calculated rates, it is natural to consider these methods as involving an approximation to χ^2. The author compares these approximations by computing examples and finds them of similar accuracy, that used with logits being better.

J. W. Tukey (Princeton, N. J.).

Johansen, N. P. Free functions. Mém. Inst. Géodésique Danemark [Geodætisk Instituts Skr.] (3) 4, 30 pp. (1944). (Danish) [MF 16270]

Let $x_1, \cdots, x_n$ be mutually independent observations with errors $\epsilon_1, \cdots, \epsilon_n$. Consider new variables y_k which are linear combinations of the x_i so that their errors satisfy the equations $\eta_i=\sum a_{ik}\epsilon_k$. The y_i are called free functions if there exist constants $q_1, \cdots, q_n$ with which $\sum_{k=1}^{n} q_k a_{ik} a_{jk}=0$ identically in i and j. It is shown that free functions have many properties in common with independent observations. In many cases functions of independent observations can be "freed" by introduction of appropriate new variables. Various relations are stated which can be useful in actual practice as exemplified by applications to geodesy.

W. Feller.

Tukey, John W. An inequality for deviations from medians. Ann. Math. Statistics 17, 75–78 (1946). [MF 15961]

If $x_1, x_2, \cdots, x_n$ are independent chance variables with zero medians, then

$$E\left|\sum_{i=1}^{n} x_i\right| \geqq (\phi(n)/n)\sum_{i=1}^{n} E|x_i|,$$

where

$$\phi(2k+1)=\phi(2k+2)=\frac{1\cdot 3\cdot 5\cdot \;\cdots\; \cdot(2k+1)}{1\cdot 2\cdot 4\cdot \;\cdots\; \cdot(2k)}.$$

When the x_i have identical distributions symmetric about the origin, the result specializes to that of Birnbaum and Zuckerman [same Ann. 15, 328–329 (1944); these Rev. 6, 160].

D. Blackwell (Washington, D. C.).

Szatrowski, Zenon. Calculating the geometric mean from a large amount of data. J. Amer. Statist. Assoc. 41, 218–220 (1946). [MF 16771]

Peters, Charles C. A new descriptive statistic: the parabolic correlation coefficient. Psychometrika 11, 57–69 (1946). [MF 15678]

The author proposes the use of a so-called parabolic correlation coefficient as a single descriptive statistic in fitting a second degree parabola to a set of data. This correlation coefficient consists of two parts, an index of the average slope and an index of curvilinearity. It is doubtful if the results offer any improvement over the usual methods of regression analysis where standard errors are derived for both the linear and quadratic coefficients.

R. L. Anderson (Raleigh, N. C.).

Jaspen, Nathan. Serial correlation. Psychometrika 11, 23–30 (1946). [MF 15676]

Following the development used in the textbooks of Kelley [Statistical Method, Macmillan, New York, 1923]

and Peters and Van Voorhis [Statistical Procedures and their Mathematical Bases, McGraw-Hill, New York, 1940] for Pearson's formula for biserial correlation, the author gives explicit formulae for triserial, quadriserial and quintiserial correlation. *C. C. Craig* (Ann Arbor, Mich.).

Delaporte, Pierre. Sur l'estimation des corrélations des caractères avec le facteur général et les facteurs de groupe et sur l'écart-type de cette estimation, en analyse factorielle. C. R. Acad. Sci. Paris **222**, 525–527 (1946). [MF 16030]

It is suggested that in estimating the correlation of a given character with the general factor in Spearman's factor-analysis it would be advantageous to modify the usual expression by weighting the ratios which enter into it by their sampling variances. Large sample approximations to these variances are then found and substituted into the expression for the estimate. *C. C. Craig.*

Lovera, G. Un'applicazione del coefficiente di correlazione alle medie statistiche. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **77**, 341–346 (1942). [MF 16253]

If n sets of variables are correlated, the average correlation coefficient is given by the usual formula for the intraclass correlation. Application to a meteorological example. *J. W. Tukey* (Princeton, N. J.).

Haldane, J. B. S. Moments of r and χ^2 for a fourfold table in the absence of association. Biometrika **33**, 231–233 (1945). [MF 16455]

Haldane, J. B. S. The use of χ^2 as a test of homogeneity in a $(n\times 2)$-fold table when expectations are small. Biometrika **33**, 234–238 (1945). [MF 16456]

★Kingston, Jorge. A Teoria da Indução Estatística. [The Theory of Statistical Induction]. Instituto Brasileiro de Geografia e Estatística, Rio de Janeiro, 1945. 121 pp. (Portuguese)

Intended to introduce teachers of classical statistics to the "modern" developments, beginning with small sample theory, this book does not go into the modern theory as far as would be suggested by the chapter headings: The problems of theoretical statistics; Significance of means and other statistics; The binomial distribution and its approximations; Lexian dispersion, contagious distributions; Chi-square and goodness of fit; Student's distribution; Estimation and comparison of variances; Analysis of variance; Testing for correlation, analysis of covariance. The level reached is typified by: (i) Student's t is introduced in the second half of the book; (ii) the analysis of variance is carried through the simplest two-way tables. The examples should be treated with caution. *J. W. Tukey.*

Brown, George W., and Tukey, John W. Some distributions of sample means. Ann. Math. Statistics **17**, 1–12 (1946). [MF 15953]

Let $S, S_0, S_1, S_2, \cdots$ be normally and independently distributed chance variables with zero means and unit variances. It is shown that

$$C_k=S\left(\prod_{i=0}^{k}S_i^{2^i}\right)^{-1},\quad P_k=\left(\prod_{i=1}^{k}S_i^{2^i}\right)^{-1}$$

are stable of index 2^{-k}. Bounds for the density function of C_k are obtained. Among other results is one which states that, if the cumulative distribution function of a chance variable has a "long tail" in a certain precisely defined sense, the distance between any two percentage points for the mean of a sample of n is ultimately larger than a positive power of n. It is claimed that these results show, among other things, that (1) "the use of the mean of a sample as a measure of location . . . implies a belief that the tails of the underlying distribution are not too long; (2) it is probable that the relative efficiencies of mean and median are greatly affected by the length of the tail." *J. Wolfowitz* (New York, N. Y.).

Halmos, Paul R. The theory of unbiased estimation. Ann. Math. Statistics **17**, 34–43 (1946). [MF 15956]

Let $F(P)$ be a real-valued function of P, where P is a probability distribution on the real line. An estimate of F from a sample of size n is a function $f(x_1, \cdots, x_n)$, where $x_1, \cdots, x_n$ make up the sample. An estimate is unbiased for a class $\{P\}$ of distributions if the expectation of f given P equals $F(P)$ for all P in $\{P\}$. The author shows that such estimates are rarely possible when a large class of P's is prescribed. (I) If $F(P)$ has an unbiased estimate, then this unbiased estimate represents it as

$$\int \varphi(x_1, \cdots, x_k)dP(x_1) \cdots dP(x_k);$$

the least value of k in such a representation is called the degree of F. (II) If $\{P\}$ contains all distributions concentrated at the numbers 0 and 1, then $\{E(x)\}^{k_1}\{E(x^2)\}^{k_2}\cdots\{E(x^n)\}^{k_n}$ is homogeneous of degree $k_1+k_2+\cdots+k_n$. (III) If $\{P\}$ contains all distributions concentrated on any finite subset of $E\subset(-\infty, +\infty)$, $F(P)$ is of degree k, and if $f(x_1, \cdots, x_n)$ is a symmetric function which is an unbiased estimate of $F(P)$, then for $x_1, \cdots, x_n \epsilon E$, f can be obtained from φ by symmetrization (and is hence unique). (IV) If symmetrization is deleted from the hypotheses of (III), then the symmetrization of f satisfies the hypotheses and conclusion of (III). (V) Among the f satisfying the hypotheses of (IV), the symmetric one has the least variance. The author points out that, while unbiased estimates are well known for $E(x^n)$, they are impossible for $E(x-\bar{x})^k$ or $\{E(x)\}^2$. An example shows that the use of an unbiased estimate may lead to practically unfortunate and pathological results. *J. W. Tukey* (Princeton, N. J.).

Tucker, Ledyard R. Maximum validity of a test with equivalent items. Psychometrika **11**, 1–13 (1946). [MF 15675]

The author points out that high item reliability may not be desirable, because high reliability does not always imply high validity for a test. The validity of a test is measured by the correlation coefficient r_{ts} between results of test T and true scores on the trait S measured. Suppose that T consists of n equivalent items (each scored 0 or 1), all of the same difficulty and the same power of discrimination, and the item-to-item reliability r_{jh} (correlation between item scores) the same for all pairs of items. When T is given to a population normally distributed on S with zero mean and unit variance, the validity of T is

$$r_{ts}=nr_{js}/(n+n(n-1)r_{jh})^{\frac{1}{2}},$$

where r_{js} is the correlation between item score and true score. Curves given for tests composed of $n=1$, 10, 100 items indicate that, when such tests have $n>1$, maximum validity occurs when item reliability r_{jh} is much less than unity, and when the item discriminative power is far from perfect. The more general and practical case where items

are not equivalent in difficulty and discrimination has not been treated. *F. Mosteller* (Princeton, N. J.).

Radhakrishna Rao, C. Information and the accuracy attainable in the estimation of statistical parameters. Bull. Calcutta Math. Soc. 37, 81–91 (1945). [MF 15683]

The author is concerned with results for finite samples, some of which are familiar as asymptotic formulas. (A) The variance of an unbiased estimate of θ is at least $1/I$, where I is R. A. Fisher's amount of information. (B) If a sufficient statistic for θ and an unbiased estimate for θ exist, then among all unbiased estimates there is one with minimum variance, which is a function of the sufficient statistic. (C) If the distribution having the sufficient statistic has the usual analytic form, there is a function of θ which has an estimate satisfying (B). These results depend on differentiation under the integral sign, and are extended to several variables. In particular, if an unbiased estimate of one parameter has minimum variance, then it is uncorrelated with any statistic whose expectation depends solely on other parameters.

The Riemannian metric $ds^2 = g_{ij}d\theta^i d\theta^j$, where

$$g_{ij} = E\left[\left(\frac{1}{\phi}\frac{\partial\phi}{\partial\theta_i}\right)\left(\frac{1}{\phi}\frac{\partial\phi}{\partial\theta_j}\right)\right],$$

and ϕ is the probability density function of the sample, is introduced into the parameter space, generating the distance proposed by Bhattacharyya [same Bull. 35, 99–109 (1943); these Rev. 6, 7]. The distance between two normal populations is calculated and a large-sample test for the equality of all parameters of two populations is proposed.

J. W. Tukey (Princeton, N. J.).

Gumbel, E. J. On the independence of the extremes in a sample. Ann. Math. Statistics 17, 78–81 (1946). [MF 15962]

It is shown that in a sample of n (large) the mth observation from one extreme and the kth from the other in order of magnitude may be regarded as independent provided m and k are small with respect to n and that the distribution function for the universe behaves in its tails in a certain exponential manner. *C. C. Craig* (Ann Arbor, Mich.).

Gumbel, Émile-J. Détermination commune des constantes dans les distributions des plus grandes valeurs. C. R. Acad. Sci. Paris 222, 34–36 (1946). [MF 15975]

The author shows how the parameters in Fréchet's distribution function for the extreme value can be estimated by means of moments of negative order. Then by means of a logarithmic transformation he shows, furthermore, that the same method can be applied in case the distribution of the extreme value has the form due to Fisher.

C. C. Craig (Ann Arbor, Mich.).

Daly, Joseph F. On the use of the sample range in an analogue of Student's t-test. Ann. Math. Statistics 17, 71–74 (1946). [MF 15960]

A proof is given that, in samples from normal, the range and the mean are statistically independent and this result is furthermore shown to be true for the mean and any symmetric function of the sample variates which is invariant under a translation of the origin. Then, in view of the independence of the mean and range, it is shown how the recent tabulation of the distribution of the range by Pearson and Hartley [Biometrika 32, 301–310 (1942); these Rev. 4, 19] makes available the analogue of the Student-Fisher t test on sample means in which the sample range is used instead of the sample standard deviation. Some results are shown which indicate that the power of this test with respect to the one-sided alternatives for which the Student-Fisher t is most powerful is little inferior for samples of 10 or less. *C. C. Craig* (Ann Arbor, Mich.).

Hoel, Paul G. Testing the homogeneity of Poisson frequencies. Ann. Math. Statistics 16, 362–368 (1945). [MF 15469]

Given two sample frequencies x and y from two independent Poisson distributions with means m_x and m_y, let $p = m_x/(m_x+m_y)$. Then, for testing the hypothesis $p = p_0$ against the alternative $p > p_0$, a common best critical region is derived. It is shown that the chi-square test with one degree of freedom is satisfactory except possibly for very small frequencies. A graph of the χ^2 critical region is given for $P[x > x_0 | p = \frac{1}{2}] = .05$, corresponding to $P[\chi^2 > \chi_0^2] = .10$.

R. L. Anderson (Raleigh, N. C.).

Walsh, John E. Some significance tests based on order statistics. Ann. Math. Statistics 17, 44–52 (1946). [MF 15957]

One-sided significance tests are developed for testing whether an observation x has the same mean as a sample $y_1, y_2, \cdots, y_m$, given that all observations are independently normally distributed with the same variance. The tests are of the form

$$x - \bar{y} > (m+1)^{\frac{1}{2}}(\bar{y} - y_u),$$

where y_u is the uth largest of the y's. Similar tests are developed for the case where the observations are sample means. If m is not too small, any desired significance level may be approximated by proper choice of u. While the tests are based on an order statistic, they are not nonparametric tests as they use the properties of the normal distribution. The advantage of the tests over the most powerful test (the t test) is in ease of computation, an advantage that might outweigh the loss of efficiency in some situations, particularly in quality control work.

The power function of the test is computed for certain values of m and compared with that of the t test. The percentage increase in sample size needed to make the test about as powerful as the t test increases as the sample size increases and is about 5% for $m=6$ and about 43% for $m=16$. *A. M. Mood* (Ames, Iowa).

Odone, Vincenzo. Il collaudo di prodotti in serie ed il calcolo delle probabilità. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. 77, 407–430 (1942). [MF 16257]

Study of acceptance sampling based on the approximation of the hypergeometric distribution by the binomial and this by the first few terms of a Gram-Charlier series. Calculation of a few examples. *J. W. Tukey* (Princeton, N. J.).

Peach, Paul, and Littauer, S. B. A note on sampling inspection. Ann. Math. Statistics 17, 81–84 (1946). [MF 15963]

A very simple approximate method of determining single sampling plans is provided. The method assumes that the defectives in a sample have a Poisson distribution, and the percentage points of chi-square are used to sum the Poisson series. *A. M. Mood* (Ames, Iowa).

Curtiss, J. H. A note on some single sampling plans requiring the inspection of a small number of items. Ann. Math. Statistics 17, 62–70 (1946). [MF 15959]

Madow, Lillian H. Systematic sampling and its relation to other sampling designs. J. Amer. Statist. Assoc. 41, 204–217 (1946). [MF 16770]

An elementary exposition of a paper by W. G. Madow and the author [Ann. Math. Statistics 15, 1–24 (1944); these Rev. 5, 210]. *J. Wolfowitz* (New York, N. Y.).

Mann, H. B. Correction to the paper "On a problem of estimation occurring in public opinion polls." Ann. Math. Statistics 17, 87–88 (1946). [MF 15965]

The paper appeared in the same Ann. 16, 85–90 (1945); these Rev. 6, 235.

Seal, H. L. The mathematics of a population composed of k stationary strata each recruited from the stratum below and supported at the lowest level by a uniform annual number of entrants. Biometrika 33, 226–230 (1945). [MF 16454]

In the problem described in the title it is assumed that the population is stationary, that a known decremental force acts which depends only on the age of an individual but not on the strata, and that the probability of an individual's moving from stratum n to stratum $n+1$ depends only on the length of time the individual has spent in the stratum. The author computes the expected numbers of entrants and of individuals for each stratum.

W. Feller (Ithaca, N. Y.).

Leslie, P. H. On the use of matrices in certain population mathematics. Biometrika 33, 183–212 (1945). [MF 16452]

The author assumes that age specific rates of fertility and of mortality are known for the population considered and that these rates remain constant over a certain period. Let P_x be the probability that a female aged x to $x+1$ at time t will be alive at time $t+1$, and F_x the number of daughters born in interval t to $t+1$, per female aged x to $x+1$ at time t, who will be alive at time $t+1$. The life table is supposed to end at age $m+1$ so that m to $m+1$ is the last age group considered. A square matrix M of $m+1$ rows is formed; the first row contains the F_x $(x=0, 1, \cdots, m)$, the subdiagonal below the principal diagonal contains the P_x $(x=0, 1, \cdots, m-1)$, and all the other elements are zero. The age distribution of the population may be represented by a one-column matrix. It is shown that the age distribution of survivors and descendants at time t may be determined by multiplying the one-column matrix corresponding to time zero by M^t. The matrix M is used to study the population, particularly its approach to a stable age distribution. *E. Lukacs* (Cincinnati, Ohio).

Del Chiaro, A. Sulla teoria formale della popolazione. Giorn. Ist. Ital. Attuari 11, 214–232 (1940). [MF 16622]

This is an expository article on the formal theory of population. The usual formulae for finding probabilities of death for a closed population are discussed. Well-known formulae, taking into account migratory movements, are derived by making appropriate assumptions regarding the mortality and the migrations. *E. Lukacs.*

Mathematical Economics

Törnqvist, Leo. On the economic theory of lottery-gambles. Skand. Aktuarietidskr. 28, 228–246 (1945). [MF 14436]

A lottery arranger A_0 has certain proceeding-parameters such as selling price of a lot, prizes to be disbursed, etc., at his disposal. The demand for lots, and consequently also the gain of A_0, are treated as chance variables the distribution of which depends on the proceeding-parameters chosen by A_0. A theory of choice of the proceeding-parameters is developed and approximate formulas for the probability distribution of the demand for lots are derived. This theory is based on the assumption that each individual tries to maximize the expected value of $U(x)$, where x denotes his gain and the function $U(x)$, called the medium-value formation function, may be interpreted as the subjective value of the gain x. *A. Wald* (New York, N. Y.).

Giguet, R. Étude de l'exploitation optimum d'un barrage en régime connu. Ann. Ponts Chaussées 1945 (115^e année), 145–172 (1945). [MF 14179]

This paper considers the economic production of hydroelectric power by a reservoir. Let $0\leqq X(t)\leqq M$ be the volume of water in the reservoir at time t, let $x(t)$ be the rate of inflow and $p(t)$ the rate of outflow, and let $y(t)$ represent general hydraulic conditions in the region. The total utility produced from $t=0$ to $t=t_0$ is $U=\int_0^{t_0}dt\int_0^{p(t)}udp$, where $u=u(p, x, y, t)$ is the instantaneous marginal utility of the outgoing water. For p not less than some fixed L, $u=0$. The functions u, x, y are supposed known, and it is desired to choose $p(t)$ so as to maximize U. Let $q(t)$ be the maximizing function. When $X(0)$ and $X(t_0)$ are given, the value of $u(q, x, y, t)$ is uniquely obtained by elementary means, and $q(t)$ is obtainable as an implicit function of u. If $x(t)$ is continuous and u obeys the law of diminishing returns with respect to p, x, and y, then $q(t)$ is continuous. During periods when $q>0$ and $0<X<M$, u is constant. The graph of u against t is composed of horizontal lines, portions of the curve $u=u(0, x, y, t)$, and portions of the curve $u=u(x, x, y, t)$. If $u(x, x, y, t)$ has a wide enough range of variation, then $q(t)$ is independent of $X(0)$ and $X(t_0)$, except during an initial and final period. Considering the present value of the marginal utility, $v=u/R_t$ where $R_t>1$ for $t>0$, does not alter the conclusion, except that, if $\int_0^{t_0}dt\int_0^{p(t)}vdp$ exists, then q does not depend on $X(\infty)$. These ideas can be applied to the general problem of savings with an upper bound to the amount that can be accumulated. It is shown that interest on the unused balance does not affect the nature of the solution if the rate of interest and the average utility of the balance are functions of t only. *H. Levene.*

Metzler, Lloyd A. Stability of multiple markets: the Hicks conditions. Econometrica 13, 277–292 (1945). [MF 14537]

The author is concerned with the types of multiple-market systems to which the Hicks stability conditions are applicable. The Hicks conditions were developed by generalizing the case of one commodity where it is assumed that an increase in price above equilibrium will create excess supply and a decrease in price will create excess demand. Depending on the behavior of other prices in the market system, Hicks gives two definitions of stability, imperfect and perfect, both of which are developed independently of

the relative speeds of adjustment in individual markets. Samuelson [Econometrica 9, 97–120 (1941); 12, 256–257 (1944)] has shown that the Hicks imperfect stability conditions are neither necessary nor sufficient, and perfect stability conditions not always sufficient, for true dynamic stability.

It is now demonstrated that (1) the Hicks perfect stability conditions are necessary, but not sufficient, conditions for stability of a market system under all possible sets of speeds of adjustment, and (2) the Hicks perfect stability conditions are both necessary and sufficient for stability of a market system when all commodities in that system are gross substitutes. The suggestion is made that the stability conditions for other market systems may also be identical with the Hicks conditions. *M. P. Stoltz.*

Haavelmo, Trygve. Multiplier effects of a balanced budget. Econometrica 13, 311–318 (1945). [MF 14538]

Let r_0 denote the individual income before a certain income tax is imposed and let r denote the corresponding gross income (including the tax) after such a tax is imposed. It is shown that, if the consumption function is linear and if total private investment is a constant, the effect of an income tax totalling T dollars that is fully spent is that the total gross income is increased by T, that is, $\sum r = \sum r_0 + T$, while total net income (gross income minus tax) and consumption remain unchanged. This result is shown to remain valid even if the consumption function is not linear, provided that the tax is proportional to income and the distribution of $r/\bar{r}$ is the same as that of $r_0/\bar{r}_0$, where $\bar{r}_0$ is the average value of r_0 and $\bar{r}$ is the average value of r.
A. Wald (New York, N. Y.).

Haberler, G. Multiplier effects of a balanced budget: Some monetary implications of Mr. Haavelmo's paper. Econometrica 14, 148–149 (1946). [MF 16294]

[Cf. the preceding review.] The author makes some comments as to the effect of a time lag between the collection and final disbursement of taxes by the government.
A. Wald (New York, N. Y.).

Goodwin, R. M. Multiplier effects of a balanced budget: The implication of a lag for Mr. Haavelmo's analysis. Econometrica 14, 150–151 (1946). [MF 16295]

[Cf. the second preceding review.] It is shown that Haavelmo's solution is reached asymptotically if a time lag between expenditure and income is assumed.
A. Wald (New York, N. Y.).

Hagen, Everett E. Multiplier effects of a balanced budget: Further analysis. Econometrica 14, 152–155 (1946). [MF 16296]

Haavelmo's analysis [cf. the third preceding review] is generalized by taking into consideration time lags and the possible effect of government spending on investment.
A. Wald (New York, N. Y.).

Haavelmo, Trygve. Multiplier effects of a balanced budget: Reply. Econometrica 14, 156–158 (1946). [MF 16297]

In reply to the comments made in the papers reviewed above, the author discusses briefly the implications of the following two assumptions: (1) consumers' expenditures depend on income during the preceding period and government expenditures are equal to taxes collected during the preceding period; (2) there is no lag between income receipts and consumers' expenditures, while government expenditures are equal to tax receipts during the preceding period. *A. Wald* (New York, N. Y.).

TOPOLOGY

Appert, Antoine. Espaces uniformes généralisés. C. R. Acad. Sci. Paris 222, 986–988 (1946). [MF 16389]

The usual axioms of uniform spaces are decomposed into a set of nine and names proposed for spaces satisfying various subsets of these. The present generalizations are the counterparts of those for topological spaces, according to which, for example, the intersection of two "open" sets need not be "open." *R. Arens* (Princeton, N. J.).

Colmez, Jean. Problème de Wiener. Recherche de solutions séparées. Caractérisation de certains espaces par leur groupe de déformations. C. R. Acad. Sci. Paris 222, 434–436 (1946). [MF 16022]

This is a continuation of earlier investigations [see Revue Sci. (Rev. Rose Illus.) 80, 313–315 (1942); these Rev. 7, 134], except that now solutions by Hausdorff topologies are sought. The term "espace séparé" is used here to designate the type of space to which the name of F. Hausdorff is usually attached. *R. Arens* (Princeton, N. J.).

Barbalat, B. Sur un groupe d'axiomes des espaces abstraits. Bull. Math. Soc. Roumaine Sci. 46, 121–133 (1944). [MF 16515]

If an operation $(\cdot)$ is defined for all subsets of a set P, where $E \subset P$ implies $E^{\cdot} \subset P$, $E^{\cdot}$ void if E is void, $x \epsilon E^{\cdot}$ exactly when $x \epsilon (E - \{x\})^{\cdot}$, then some of the following may be satisfied: (1) if $E \subset F$ then $E^{\cdot} \subset F^{\cdot}$, (2) $(E \cup F)^{\cdot} \subset E^{\cdot} \cup F^{\cdot}$, (3) $\{x\}^{\cdot}$ is void, (4) $(E \cup E^{\cdot})^{\cdot} \subset E \cup E^{\cdot}$, (5) $E^{\cdot\cdot} \subset E^{\cdot}$. If by 134, for example, we mean that 1, 3, and 4, but not 2 and 5, are satisfied, then 1234, 1235, 125, 235, 25 are said to be impossible, and the other twenty-seven combinations are shown to be possible. *R. Arens* (Princeton, N. J.).

de Groot, J. Topological classification of all closed countable and continuous classification of all countable pointsets. Nederl. Akad. Wetensch., Proc. 48, 237–248 = Indagationes Math. 7, 42–53 (1945). [MF 15796]

Closed countable subsets of the real number system are shown to be homeomorphic if and only if they have the same "degree," where for a compact set A the degree is defined thus. Let $A = A_0$; for any ordinal $\beta > 0$ let A_β be the derived set of the intersection A^β of all A_α, $\alpha < \beta$. Let A_α be the first A_β which is void (which must exist since no A_β is perfect); then A^α has a finite number n of points. The degree is the pair (α, n), and, in fact (this is attributed to H. Freudenthal), A is homeomorphic to the ordinal $\omega^\alpha n + 1$. For closed sets A, the degree is (β, α, n), where β is the first ordinal for which A_β is compact, and (α, n) is the degree of A_β. A comparison of the degrees of closed countable sets shows also whether one may be mapped continuously onto the other. Finally, there are shown to be $\mathfrak{c}$ nonhomeomorphic countable sets. *R. Arens* (Princeton, N. J.).

de Groot, J. Some topological problems. Nederl. Akad. Wetensch., Proc. 49, 47–53 = Indagationes Math. 8, 11–17 (1946). [MF 16341]

This paper states without proofs a variety of results, some

of which are proved in the author's dissertation [Topologische Studien, University of Groningen, 1942; these Rev. **7**, 135] and in the paper reviewed above. A new result on the extension of mappings is as follows. Consider two compact (separable) spaces X and Y, with a closed continuous mapping f of a set $A\subset X$ on a set $B\subset Y$. Suppose $Y-B$ contains no continuum, and suppose each $x\varepsilon X-A$ has a basis $\mathfrak{U}$ such that, if two subsets S and T lie in $U\cap A$, $U\varepsilon\mathfrak{U}$, and have x as a common limit point, then any subset of U which is open and closed in U and contains S must meet T. Then f can be extended to a (closed) mapping of X in Y.
R. Arens (Princeton, N. J.).

de Groot, J. Continuous classification of all microcompact 0-dimensional spaces. Nederl. Akad. Wetensch., Proc. **49**, 518–523 = Indagationes Math. **8**, 337–342 (1946).

Continuing investigations begun in the two papers reviewed above, the author establishes a complete list of those properties of locally compact separable 0-dimensional metric sets which are preserved under continuous mapping, exclusive of properties common to all such sets. It is pointed out why the unsolved problem of topologically classifying these sets is more difficult. *R. Arens* (Princeton, N. J.).

Yoshizawa, Hisa-aki. On simultaneous extension of continuous functions. Proc. Imp. Acad. Tokyo **20**, 653–654 (1944). [MF 14938]

Suppose that Ω is a bicompact Hausdorff space and F is a closed subset of Ω. Let $C(\Omega)$ and $C(F)$ be the normed rings of all complex-valued continuous functions on Ω and F, respectively. Suppose that there exists a continuous mapping of Ω on F, $\omega'=f(\omega)$, such that $f(\omega')=\omega'$ for all $\omega'\varepsilon F$ (in other words, a retract in the sense of Borsuk). Then, letting $x'(\omega')$ be arbitrary in $C(F)$, it is clear that the correspondence $x'(\omega')\rightarrow x(\omega)=x'[f(\omega)]$ is a mapping of $C(F)$ on a subring of $C(\Omega)$ which is (1) linear:

$$x'(\omega')+y'(\omega')\rightarrow x(\omega)+y(\omega);$$

(2) multiplicative: $x'(\omega')\cdot y'(\omega')\rightarrow x(\omega)\cdot y(\omega)$; (3) isometric: $|x'-y'|=|x-y|$. Note that $x(\omega)$ is an extension of $x'(\omega')$ since $x(\omega')=x'(\omega')$ for all $\omega'\varepsilon F$. The purpose of this paper is to prove the converse of this, namely, a mapping of $C(F)$ on a subring of $C(\Omega)$ which is an extension satisfying (1), (2) and (3) is generated by a retract of Ω on F.
E. R. Lorch (New York, N. Y.).

Ore, Oystein. Mappings of closure relations. Ann. of Math. (2) **47**, 56–72 (1946). [MF 15659]

The author's introduction is as follows. The present paper contains a discussion of some general properties of mappings of closure relations and particularly of the semi-continuous mappings of a closure space into ordered sets. The main problem is a so-called Galois problem for mappings and its solution is obtained for the family of all semi-continuous order functions of the space. A characterization of the system of all functions with values in ordered sets may also be of some interest. *S. Eilenberg* (Bloomington, Ind.).

Pétresco, Julien. Sur les sommes et les intersections d'ensembles des espaces topologiques. Mathematica, Timişoara **21**, 84–94 (1945). [MF 13974]

The author's interest is in enhanced logical refinement and precision of definitions in set theory. For instance, according to the author: to "define," "mod. set theory," that is, from the point of view of set theory, a subset A of the given set E is to "know the values [true or false] of the logical function $x\varepsilon A$, for every $x\varepsilon E$." Again, a topological property τ of (or at) the point x, in relation to a given set A, say, as is usually the case (we denote this dependence of τ on A by writing τ_A), is defined by a logical function expressed in terms of logical functions defining, mod. set theory, neighborhoods $V(x)$ of x, the reference being to a Hausdorff topological space satisfying his first three axioms. To "define" the set A topologically (mod. τ) is to "know" the values of the logical function (of x): $\tau_A(x)$: "x has property τ in relation to the set A," for all $x\varepsilon E$. The author considers various elementary topological properties, such as $\tau=\mathrm{Int}$ (x is an interior point of A), $\tau=\mathrm{Fr}$ (x is a frontier point of A), etc., and shows, for example, that A is defined topologically if it is defined (mod. Fr). The main problem taken up is the topological definition of the sum and product of sets (elements of an aggregate of arbitrary cardinal) in terms of the topological definition of these sets. The discussion utilizes Hausdorff's lim sup of an infinite sequence of sets. Applications are indicated to local connectivity and Borelian systems. *H. Blumberg* (Columbus, Ohio).

Jones, F. Burton. Concerning the separability of certain locally connected metric spaces. Bull. Amer. Math. Soc. **52**, 303–306 (1946). [MF 16194]

A space S is bicyclic if it is cut by no pair of its points. The author shows, using a transfinite induction argument, that every bicyclic locally connected complete metric space S which does not contain uncountably many primitive skew curves of type 1 is separable. However, there exist bicyclic locally connected nonseparable complete Moore spaces containing no primitive skew curves of type 1. If M is a bicyclic locally connected complete Moore space containing no primitive skew curve of type 1, then in order that M be metric it is necessary and sufficient that M be completely (perfectly) separable. Using this result it is shown that every metric space satisfying axioms 0–4 of R. L. Moore's "Foundations of Point Set Theory" [Amer. Math. Soc. Colloquium Publ., v. 13, New York, 1932] is completely (perfectly) separable. *D. W. Hall.*

Pepper, Paul M. Concerning pointwise-symmetry. Rep. Math. Colloquium (2) **7**, 29–36 (1946). [MF 16132]

Call a mapping f of a metric space M on itself an η-symmetry with center p, $0<\eta\leqq 1$, if $f(p)=p$ and $xf(x)\geqq 2\eta xp=2\eta pf(x)$. For each $\eta<1$ there are bounded metric spaces with arbitrarily small diameter and at least two η-symmetries with different centers. However, if M is bounded, there is at most one center for 1-symmetries. (~~This holds even when M is only semimetric and xf(x)=2ηp =2ηpf(x).)~~ If M is bounded and metric there is at most one point p with these properties: for every $\epsilon>0$ there is a mapping f_ϵ of M on itself with $xf_\epsilon(x)\geqq 2xp-\epsilon$ and $|f_\epsilon(x)p-xp|\leqq\epsilon$. *H. Busemann* (Northampton, Mass.).

Denjoy, Arnaud. Topologie des espaces cartésiens. C. R. Acad. Sci. Paris **222**, 28–31 (1946). [MF 15974]

The author defines a normal decomposition of r-dimensional Euclidean space U_r as a division of U_r into disjoint sets, each of which is either a point or a bounded continuum. [The use of the word "continuum" is not specified.] Sets of accumulation, sets of convergence and limit sets are defined for normal decompositions in terms of a well-known topology for subsets of a space. Various general results about convergence of sequences in a normal decomposition are established. *E. Hewitt* (Bryn Mawr, Pa.).

Rodnyansky, A. M. Irreducible continua and local connectedness. C. R. (Doklady) Acad. Sci. URSS (N.S.) 49, 83–84 (1945). [MF 16400]

The author states some implications of local connectedness in a metric separable space C which is either (a) irreducibly connected between two points or (b) an irreducible (compact) continuum about two points. In either case, local connectedness at a single point implies the existence of "small" connected open sets about the point. Hypothesis (a) together with local connectedness or local compactness at every point implies that C is an arc. [The first of these results is known. See G. T. Whyburn, Bull. Amer. Math. Soc. 33, 685–689 (1927); Trans. Amer. Math. Soc. 32, 926–943 (1930).] Under hypothesis (b) the components of the set M of points of nonlocal connectedness are either continua or semicontinua which with the adjunction of one or two points become continua. All possible topological types of components of $C-M$ are listed. Some results are stated concerning the boundary of components of $C-M$, and the possible cardinal numbers of the sets of all components of various types of M and $C-M$ are given.

J. L. Kelley (Chicago, Ill.).

Whyburn, G. T. On monotone retractability into simple arcs. Bull. Amer. Math. Soc. 52, 109–112 (1946). [MF 15449]

With reference to locally connected continua A, consider the properties (π) every simple arc in A is a monotone retract of A and (II) every monotone image of A has property (π). The author shows that (1) for locally connected continua in general, (II) is equivalent to unicoherence, (2) for plane locally connected continua, (π) is equivalent to unicoherence and (3) every closed 2-dimensional connected manifold has property (π). *W. W. S. Claytor.*

Wojdyslawski, M. Quelques applications d'un critérium pour qu'un continu soit plan. Rec. Math. [Mat. Sbornik] N.S. 18(60), 29–40 (1946). (Russian. French summary) [MF 16675]

The summary, which gives a good idea of the contents of the paper, is as follows.

En m'appuyant sur un théorème de M. Claytor [Ann. of Math. (2) 35, 809–835 (1934)] je donne ici (§ 1) premièrement un critérium pour qu'un continu localement connexe forme un ensemble homéomorphe à un ensemble plan. Les autres paragraphes contiennent quelques applications de ce critérium.

Dans le § 2 je m'occupe de la quasi-homéomorphie. Les continus X et Y s'appellent quasi-homéomorphes, s'il existe pour chaque $\epsilon>0$ une transformation de l'un en l'autre à tranches d'un diamètre $<\epsilon$. Le théorème 3 donne une caractérisation topologique des continus quasi-homéomorphes à un cercle plan et le théorème 4 énonce qu'un ensemble quasi-homéomorphe à une multiplicité à deux dimensions lui est homéomorphe. C'est une solution positive pour la dimension 2 d'un problème de MM. S. Ulam et C. Kuratowski [Fund. Math. 20, 244–253 (1933)].

Le § 3 est consacré à l'asphéricité. Un continu X s'appelle asphérique s'il n'existe aucune transformation essentielle de la sphère m-dimensionnelle pour $m\geqq 2$ en X. J'établis que chaque ensemble plan ou bien 1-dimensionnel est asphérique, ce qui était démontré par M. W. Hurewicz [Nederl. Akad. Wetensch., Proc. 38, 112–119, 521–528 (1935); 39, 117–126, 215–224 (1936)] sous la condition supplementaire d'être un rétracte de voisinage.

Enfin, je montre dans le § 4 qu'une multiplicité à trois dimensions peut être représentée au plus d'une manière comme le produit cartésien de deux facteurs; l'un se trouve être une circonférence et l'autre une multiplicité à deux dimensions déterminée d'une façon univoque.

L. Zippin (Flushing, N. Y.).

Haupt, Otto. Limessätze bei geometrischen Ordnungen. Ann. Mat. Pura Appl. (4) 23, 123–148 (1944). [MF 16613]

The author considers a compact continuum K in Euclidean space R^n. Properties of K are studied by means of order-numbers, obtained from the number of components (or the number of points) of KE^{n-1}, where E^{n-1} is a hyperplane in R^n. An f-system of hyperplanes with respect to K is defined by the property that, for every hyperplane E in f such that $KE\neq 0$, there exist two hyperplanes E' and E'' in f, arbitrarily close to E, and such that (1) $K(EE'+EE'')=0$, and (2) E' and E'' determine two angle-spaces with arbitrarily small angle, whose sum contains KE. In particular, the set of all hyperplanes is an f-system. The strong $(K|f)$-order of K (respectively, the strong $(P|f)$-order of K) is defined as the maximum of the number of components (respectively, the number of points) in KE^{n-1} for all E^{n-1} in f. In all cases of interest these orders are finite. The terms weak $(K|f)$-order and weak $(P|f)$-order are defined as above except that any subset of f which is nowhere dense in f may be excluded. Theorem: if the continuum K has weak $(K|f)$-order equal to k then K has strong $(K|f)$-order not exceeding $2k$.

The principal results of the paper are as follows. Suppose the sequence K_i of uniformly bounded continua converges to the continuum K. If f is any f-system of hyperplanes and K_i has weak $(K|f)$-order not exceeding k, for every i, then K has weak $(K|f)$-order not exceeding k. If K_i has weak $(P|f)$-order not exceeding k for every i, and f is the set of all hyperplanes, then K has weak $(P|f)$-order not exceeding k. *J. H. Roberts* (Durham, N. C.).

Leray, Jean. Sur la forme des espaces topologiques et sur les points fixes des représentations. J. Math. Pures Appl. (9) 24, 95–167 (1945). [MF 15970]

Leray, Jean. Sur la position d'un ensemble fermé de points d'un espace topologique. J. Math. Pures Appl. (9) 24, 169–199 (1945). [MF 15971]

Leray, Jean. Sur les équations et les transformations. J. Math. Pures Appl. (9) 24, 201–248 (1946). [MF 15972]

The three papers constitute the main part of a course in algebraic topology aiming toward a direct approach to problems concerning the existence of solutions of equations, which are the author's main interest. The starting remark is that the product of two abstract complexes is again an abstract complex, while the product of two simplicial complexes is not (without further construction) a simplicial complex. In order to take full advantage of this convenient property of abstract complexes the author by-passes the usual "covering-nerve" scheme and constructs a cohomology theory of topological spaces more directly connected with abstract complexes. The resulting cohomology theory seems to be isomorphic (at least for compact Hausdorff spaces) with the groups obtained following the Čech scheme.

The abstract complexes C used are finite. If the complex has only cells of positive dimension and if the 0-cochain C^0 which has the value 1 on every 0-cell is a cocycle, then C is

said to have a unit and C^0 is called the unit of C. A concrete complex K consists of an abstract complex C, a topological space X and a function which to every q-cell σ^q of C assigns a nonvacuous subset $|\sigma^q|$ of X (called the support of σ^q), subject to the condition that $|\sigma^{q+1}| \subset |\sigma^q|$ whenever σ^q and σ^{q+1} are incident. Whenever in constructions the supports become empty, the complex C is automatically restricted to the subcomplex whose cells have nonempty supports. If $K_1=(C_1, X_1)$ and $K_2=(C_2, X_2)$ are concrete complexes then $K_1 \times K_2 = (C_1 \times C_2, X_1 \times X_2)$ is a concrete complex with supports $|\sigma^p \times \tau^q| = |\sigma^p| \times |\tau^q|$. If $K_1=(C_1, X)$ and $K_2=(C_2, X)$ then $K_1 . K_2 = (C_1 \times C_2, X)$ with supports

$$|\sigma^p \times \tau^q| = |\sigma^p| \cap |\tau^q|.$$

If $K=(C, X)$ and $Y \subset X$, then $K.Y=(C, Y)$ with supports $|\sigma^q| \cap Y$. If $f: Y \to X$ is a continuous map and $K=(C, X)$ then $f^{-1}K=(C, Y)$ with supports $f^{-1}(|\sigma^q|)$. A concrete complex $K=(C, X)$ is called a cover of X if (1) the complex C has a unit, (2) the supports $|\sigma^q|$ are closed and cover X, (3) for every $x \epsilon X$ the concrete complex $K.x$ has the cohomology groups of a point. The family of covers is closed under all the operations defined above. Every finite covering of the space X by closed sets gives a cover, with the nerve as the abstract complex and the various intersections of the closed sets as supports.

A cochain L (with coefficients in some ring with a unit) of a space X is by definition a cochain in any cover K of X. The operations $L_1 \times L_2$, $L_1 . L_2$, $L.Y$ and $f^{-1}(L)$ are then suitably defined. If L is a cochain in the cover K and L_1^0 is the unit cochain of a cover K_1 then the following identifications are made: $L = L_1^0 . L = L . L_1^0$. Using these cochains the cohomology groups and the cohomology ring of any space X are defined in the usual way.

Chapter I is devoted to these definitions and to proving the basic properties of the cohomology groups. As an illustration of the technique H. Hopf's theorems concerning the ring of a group manifold [Ann. of Math. (2) **42**, 22–52 (1941); these Rev. **3**, 61] are generalized to compact Hausdorff spaces. The same generalization has been made previously with equal ease by S. Lefschetz [Algebraic Topology, Amer. Math. Soc. Colloquium Publ., vol. 27, New York, 1942, pp. 335–341; these Rev. **4**, 84] using the Čech groups.

In chapter II the cohomology rings are effectively calculated for compact Hausdorff spaces which have a convexoid covering, that is, a finite covering by closed sets each of which together with each of their nonvacuous intersections has the homology groups of a point. As a tool the theory of dual complexes is developed very much as by Lefschetz [op. cit.].

In chapter III the Lefschetz number Λ_ξ of a transformation $\xi: X \to X$ of a compact Hausdorff space having a convexoid covering is defined using the same method as Lefschetz [op. cit., p. 191]. It is then proved under additional local assumptions on X that if $\Lambda_\xi \neq 0$ then ξ has a fixed point. The local assumptions are very strong and are not satisfied by absolute neighborhood retracts, although they are satisfied by polyhedra and other spaces occurring in analysis.

In chapter IV the cohomology ring modulo a subset is defined and studied. As an application the Mayer-Vietoris formulae are established for the cohomology rings. A new ring of "pseudocycles" in a space is defined by considering the inverse limit of the cohomology rings of the compact Hausdorff subspaces. Chapter V is devoted to differentiable manifolds. The usual duality theorems are stated. Some of the concepts (for example, "convex cell") are undefined, so that the reviewer was unable to follow the arguments.

The third paper (chapters VI–VIII) is the crux of the theory. The author develops systematically the topological theory of equations, thus continuing his earlier work with J. Schauder [Ann. Sci. École Norm. Sup. (3) **51**, 45–78 (1934)]. Let E be a compact Hausdorff space satisfying the local conditions of chapter III and let ξ be a continuous mapping of a closed subset of E into E. Given an open set O in E on which ξ is defined and such that the equation $\xi(x)=x$ has no solution on the boundary of O, the author defines the integer $i(O)$ which is the algebraic number of solutions of the equation $\xi(x)=x$ in O. The definition of $i(O)$ is similar to that of the Lefschetz number and in fact $i(O)=\Lambda_\xi$ if $O=E$. Numerous properties of $i(O)$ are derived. More complicated equations depending on additional parameters are similarly discussed. For the actual applications to analysis the reader is referred to earlier papers.

S. Eilenberg (Bloomington, Ind.).

Lauritzen, Svend. On mapping of a nonorientable surface on itself. Mat. Tidsskr. B. **1946**, 92–96 (1946). (Danish) [MF 16310]

If φ is a nonorientable, compact, connected, 2-dimensional manifold whose genus k and number q of boundaries satisfy the condition $k+q>2$, then the universal covering surface of φ may be represented as the "convex region" K_φ of a non-Abelian group of linear substitutions whose elements are of two types: $w=(az+b)/(\bar{b}z+\bar{a})$, $a\bar{a}-b\bar{b}>0$, $(a-\bar{a})^2+4b\bar{b}>0$ (substitution of the first kind), and $w=(c\bar{z}+d)/(\bar{d}\bar{z}+\bar{c})$, $c\bar{c}-d\bar{d}>0$, $d+\bar{d}\neq 0$ (substitution of the second kind). Just as in the classical case (where φ is orientable and the substitutions are of the first kind) a topological mapping of φ is covered by a homeomorphism $\mathfrak{K}$ of K_φ on itself which induces an automorphism I of the fundamental group F and can be extended to a homeomorphism of $\bar{K}_\varphi$ [cf. J. Nielsen, Acta Math. **50**, 189–358 (1927); **53**, 1–76 (1929)].

R. H. Fox (Princeton, N. J.).

Bundgaard, Svend, and Nielsen, Jakob. Unified proofs of some theorems in surface topology. Mat. Tidsskr. B. **1946**, 1–16 (1946). (Danish) [MF 16305]

Starting out from several simple lemmas and a preliminary discussion of the homeomorphisms of a simple closed curve on itself, the authors give rather easy proofs of selected theorems about the effect of a surface transformation on the elements of the fundamental group, on the points of the universal covering space and on the points of the limit circle. The theorems are implicitly contained in earlier results of Nielsen [Acta Math. **53**, 1–76 (1929); **58**, 87–167 (1932)] which are at the same time more delicate and more difficult to prove.

R. H. Fox (Princeton, N. J.).

Nielsen, Jakob. Surface transformation classes of algebraically finite type. Danske Vid. Selsk. Math.-Phys. Medd. **21**, no. 2, 89 pp. (1944). [MF 13894]

A surface transformation is an orientation-preserving homeomorphism of a compact, connected, orientable, 2-dimensional manifold (which may or may not have boundaries). Transformations belong to the same class when they are homotopic; a class of surface transformations is of finite order n if a representative transformation is homotopic to its nth iterate. A fundamental result, established by the author in a previous paper [Acta Math. **75**, 23–115 (1943); these Rev. **7**, 137], is that a class of order n always contains a transformation which is periodic of period n.

The present memoir is concerned with a rather far-reaching generalization of transformation classes of finite order for which the author proposes the term algebraically finite classes. Roughly speaking, a class is algebraically finite if the surface can be decomposed by a set of disjoint simple closed curves into a finite number of parts S_i such that on each part the class is of finite order, that is, if attention is fixed on any single S_i, a transformation representing the class can be so chosen that it maps S_i upon itself and is periodic on S_i with some finite period n_i. Many classes which are not of finite order are algebraically finite: for example, cut one of the "handles" of a closed surface of genus p, represented as a "sphere with p handles," and match the cut edges after rotating one of the cuts through a certain number of complete revolutions. The classes of algebraically finite type have in common with the classes of finite order that the roots of the characteristic polynomial (of the automorphism induced in the 1-dimensional homology group) are all roots of unity. The author conjectures that this property of the characteristic polynomial is possessed by (those and) only those classes which are of algebraically finite type. In fact, it is stated that this conjecture is the motivation for the selection of the term algebraically finite type.

The memoir is divided into three parts. Part I outlines without proofs the general Nielsen method of treating surface transformations by investigating their effect on the universal covering surface and especially on the set of limit points [cf. the paper cited above]. The exact definition of class of algebraically finite type is made in terms of these concepts and is too complicated to state here. It should be noted that this method applies only to surfaces of non-negative Euler characteristic. However, of the four surfaces excluded, three (the sphere, the 2-cell, and the circular ring) have no transformation classes which are not of finite order while the complete set of transformation classes of the torus, not all of which are of finite order, may easily be enumerated and examined separately.

The principal result of part II is that any class of algebraically finite type is represented by a transformation which has no fixed points of index 0 and whose fixed points of index different from 0 all represent different classes of fixed points. The proof of this theorem makes heavy use of the principal result of the paper cited above. Necessary and sufficient conditions are given for the topological equivalence of classes of surface transformations of algebraically finite type. Necessary and sufficient conditions for such equivalence had previously been established by the author [same Medd. 15, no. 1 (1937)] for classes of finite order.

In part III the characteristic polynomial of a class of algebraically finite type is explicitly calculated, showing in particular that its roots are all roots of unity. The polynomial exhibited also has meaning as regards the various classes of fixed points. *R. H. Fox* (Princeton, N. J.).

Komatu, Atuo. Zur Topologie der Abbildungen von Komplexen. Jap. J. Math. 17, 201–228 (1941). [MF 14957]

Theorems about the obstacle cocycle [cf. Eilenberg, Ann. of Math. (2) 41, 231–251 (1940); these Rev. 1, 222] are generalized in § 3 to the case where the image complex is not required to be simple. In this general case the obstacle becomes a cocycle, not in the antecedent complex, but in a certain covering of it. These results are applied in § 4 to show that the existence of an essential mapping into an n-dimensional manifold is equivalent to the existence of a nontrivial n-dimensional cocycle in some covering and in § 5 to show that an n-dimensional complex which is stable in the large has a nontrivial n-dimensional cycle in some covering. [The results of these two sections are probably correct. However, the proofs make essential use of a certain group $\mathfrak{A}$ whose definition is incoherent (the definition of $\mathfrak{A}$ seems to require one to add elements $k(\alpha)$ and β which do not belong to the same group).] In § 1 well-known theorems about the lifting of mappings to covering spaces are proved. The groups ν, μ, and λ, considered in § 2, though perhaps novel in 1940, are well known nowadays as the kernel-image subgroups of the homotopy sequence $\pi_n(K^n)\to\pi_n(K^n/K^{n-1})\to\pi_{n-1}(K^{n-1})$. In an appendix there is given an example of a simply connected r-dimensional pseudomanifold which can be mapped onto the r-sphere with degree 2 but not with degree 1. Another example shows that a mapping of one pseudomanifold on another may map a contractible set onto the carrier of a nontrivial cycle. *R. H. Fox.*

Chern, Shiing-shen. Characteristic classes of Hermitian manifolds. Ann. of Math. (2) 47, 85–121 (1946). [MF 15661]

The object of this paper is twofold: first, to extend known results concerning sphere bundles over a differentiable manifold M and the characteristic cohomology classes of M to complex sphere bundles and characteristic classes of a complex manifold; second, to show that, in case a complex manifold M carries a Hermitian metric, the characteristic classes of M are completely determined by the local geometry, the basic characteristic classes being expressible by exact exterior differential forms constructible from the metric.

Let M be a complex manifold of (complex) dimension n (topological dimension $2n$). The tangent vectors of M define a complex sphere bundle B over M, that is, a fibre bundle such that the fibres are homeomorphic to the complex n-sphere and the group in each fibre is the unitary group $U(n)$. Then B determines a mapping T of M into the Grassmann manifold $H(n, N)$ of all n-dimensional linear subspaces of a complex vector space $E(n+N)$ of complex dimension $n+N$, $N\geqq n$. A cohomology class W of M which is the inverse image under T of a cohomology class of $H(n, N)$ is called a characteristic class. The author defines a set of invariant exact exterior differential forms Φ_r, $r=1, \cdots, n$, of $H(n, N)$, such that every such form of degree not greater than $2n$ is a polynomial in Φ_r (with constant coefficients). If a cohomology class of $H(n, N)$ contains the cocycle of topological dimension $2(n-r+1)$ defined by Φ_r, its inverse image W_r under T is called a basic characteristic class; all characteristic classes can be obtained from these basic classes by operations of the cohomology ring. These basic classes are identified with the classes obtained by generalizing to complex manifolds the well-known classes of Stiefel and Whitney.

If M is a compact complex manifold carrying an intrinsic Hermitian differential form $ds^2=g_{ij}(z, \bar{z})dz^i d\bar{z}^j$ ($\bar{g}_{ij}=g_{ji}$), the n basic classes W_r are expressible in the sense of de Rham by differential forms Ψ_r which can be constructed directly from the Hermitian metric. For $r=1$, the relation between the characteristic class and the corresponding differential form reduces to the Allendoerfer-Weil generalization of the Gauss-Bonnet theorem if we interpret the Hermitian metric as a Riemannian metric of a real $2n$-manifold. Finally, the author shows that it is possible to define in complex projective space an elliptic Hermitian geometry and uses the

preceding general results to obtain theorems generalizing classical formulas of Cartan and Wirtinger.

S. B. Myers (Ann Arbor, Mich.).

Montgomery, Deane, and Samelson, Hans. Fiberings with singularities. Duke Math. J. 13, 51–56 (1946). [MF 15873]

A map $f:R\rightarrow B$ is called a fibering with singularities if (1) f is open, (2) for a closed set L in R, $f|L$ is topological, and (3) $R-L$ is a fiber bundle over $B-f(L)$. The principal results are confined to the case where R is an n-sphere, $B-f(L)$ is a differentiable manifold, f is differentiable and the fibers F are differentiable manifolds differentiably imbedded in $R-L$. It is proved that L is not a point. If in addition the Euler number of F is zero, and L consists of a finite number k of points, then $k=2$ if n is even, and $k=0$ if n is odd. These and additional results are relevant to two conjectures: (1) Euclidean n-space cannot be fibered with a compact fiber, (2) if a Lie group of transformations of an n-sphere has one fixed point, it must have two.

N. E. Steenrod (Ann Arbor, Mich.).

Rey Pastor, Julio. The last geometric theorems of Poincaré and their applications. Union Mat. Argentina. Memorias y Monografias (2) 1, no. 4, 42 pp. (1945). (Spanish. French summary) [MF 14552]

The author gives a proof of the theorem on existence of a fixed point under a homeomorphism of an annulus on itself and compares it with previous proofs. The theorem is generalized by replacing the condition that points on the two boundary circles are moved in opposite directions by a weaker one. Applications are given to the billiard ball problem. [See G. D. Birkhoff, Dynamical Systems, Amer. Math. Soc. Colloquium Publ., vol. 9, New York, 1927, chap. VI.]

W. Kaplan (Ann Arbor, Mich.).

GEOMETRY

Erdös, P. On sets of distances of n points. Amer. Math. Monthly 53, 248–250 (1946). [MF 16448]

For a set of n points in the plane, estimates are given for (1) the minimum number of different distances, (2) the maximum number of times a given distance can occur, (3) the maximum number of times that the maximum and minimum distances can occur.

I. Kaplansky.

Gentile, Giovanni. Punti diagonali e poligoni di divisione di un n-gono piano convesso. Boll. Mat. (4) 1, 71–74 (1940). [MF 16543]

The author proves some known elementary results concerning the configuration formed by a convex polygon and its diagonals. He corrects an erroneous statement of Sapienza [same Boll. (3) 17, 119–123 (1938)].

I. Kaplansky.

Labra, Manuel. Calculation of the sides of regular inscribed polygons. Revista Soc. Cubana Ci. Fís. Mat. 2, 47–67 (1945). (Spanish) [MF 16907]

Fog, David. A theorem on four circles. Mat. Tidsskr. B. 1946, 113–119 (1946). (Danish) [MF 16312]

In the Euclidean plane consider four oriented circles c_1, c_2, c_3, c_4. Each pair of circles c_i, c_k has one pair of common oriented tangents. Denote by $t_{ki}=t_{ik}$ the length of the segments intercepted on these tangents by the points of contact with c_i and c_k. There are two oriented circles that touch c_4, c_2, c_1 (or c_4, c_2, c_3). Let P and Q (U and V) be the points of contact of these circles with c_2, where P (U) belongs to the circle whose orientation goes from c_4 over c_2 to c_1 (from c_4 over c_2 to c_3). Then

$$t_{13}^2t_{24}^2=(t_{12}t_{34}+t_{23}t_{14})^2-4t_{12}t_{23}t_{34}t_{14}(PUVQ),$$

where $(PUVQ)$ is the cross ratio of the four points on c_2 and equals the correspondingly defined cross ratios on c_1, c_3, c_4. An analogous theorem for spherical geometry is derived.

H. Busemann (Northampton, Mass.).

Klíma, J. On some motions of a variable figure in the plane. Věstník Královské České Společnosti Nauk. Třída Matemat.-Přírodověd. 1944, 5 pp. (1944). (Czech) [MF 16367]

The paper contains purely synthetic proofs of two theorems of Study [see Study, Vorlesungen über Ausgewählte Gegenstände der Geometrie, vol. I, Teubner, Leipzig-Berlin, 1911, p. 16] and generalizations. Theorem 1: if two opposite vertices of a square perform uniform motions on straight lines so do the other two vertices. Theorem 2: if two opposite vertices of a square move on two circles with the same angular velocity and in the same sense so do the other two vertices.

C. Loewner (Syracuse, N. Y.).

Bone, H. B. On orthogonal conic sections. Mathematica, Zutphen. B. 11, 132–150 (1943). (Dutch) [MF 15728]

Two conic sections are called orthogonal if they cut orthogonally in each point of intersection. The conic sections orthogonal to a given conic section K are, first of all, the confocal conic sections; one of them is K itself, the points of intersection then being the four points S_i of K having isotropic tangent lines. By means of the methods of analytic geometry the author proves that the following sets are orthogonal to a given general conic section K. (1) The confocal conic sections. (2) Three pencils L_i of conic sections; a conic section of L_i cuts K in four different points. (3) All double lines of the plane. (4) The pairs of normals of K which intersect in a point of K. (5) Four sets of conic sections, each containing conic sections which are tangent to K in a point S_i. (6) Six families of double-tangent conic sections, the tangent points being S_i, S_j. (7) Four sets of conic sections osculating in S_i. (8) Four sets of conic sections hyperosculating in S_i.

O. Bottema (Delft).

Soloviov, P. Sur un problème géométrographique lié au tracé de la parabole. Nauk.-Doslid. Inst. Mat. Meh. Harkiv. Univ. Geometričnii Zbirnik 2, 145–151 (1940). (Ukrainian. French summary) [MF 16952]

Solution du problème: trouver sur l'axe de la parabole un point $C(x_0, 0)$ tel que la valeur maximum de sec α soit la plus petite, α étant l'angle que fait le rayon vecteur CM de C au point M de la parabole avec la normale à la parabole en M.

Author's summary.

Baron, H. J. Die Ankugeln des Tetraeders in Beziehung zur Umkugel. Tôhoku Math. J. 48, 185–192 (1941). [MF 16358]

The author considers upper limits for the radii $\rho_a\leqq\rho_b\leqq\rho_c\leqq\rho_d$ of the escribed spheres for tetrahedra having a circumsphere of given radius r. These are $\rho_a/r\leqq 2/3$, $\rho_b/r\leqq(2\sqrt{3}-3)^{\frac{1}{2}}$, $\rho_c/r\leqq 4\sqrt{3}/9$, $\rho_d/r\leqq 1$. Certain limiting cases are also considered.

J. S. Frame.

Court, N. A. A skew quartic associated with a tetrahedron. Duke Math. J. 13, 123–128 (1946). [MF 15881]

The locus of the line of intersection of two variable orthogonal planes revolving about two fixed skew axes a and a' is the orthogonal hyperboloid of one sheet (aa'). Consider a tetrahedron $T=ABCD$, $BC=a$, $DA=a'$, $CA=b$, etc. The hyperboloids (aa'), (bb') and (cc') belong to the same pencil of quadrics; they have in common a skew quartic C_4, associated with the tetrahedron; C_4 passes through the vertices of T; it is the locus of points from which the three pairs of opposite edges of T are projected by three pairs of orthogonal planes. A tetrahedron T is transformed into an orthocentric tetrahedron by polar reciprocation with respect to a sphere if, and only if, the center of the sphere lies on C_4. The planes perpendicular at a point M to the lines joining M to the vertices of T meet the respective faces of T in four lines which are either coplanar or form a hyperbolic group, according as M does or does not lie on C_4. The quartic C_4 lies on the hyperboloid H determined by the altitudes of T and passes through the feet of the altitudes. The tangent planes to H at the vertices of T are the osculating planes, at the respective points, of C_4.

O. Bottema (Delft).

Bujmola, G. Ueber die meist gebrauchten Zeicheninstrumente und über Symbole der damit ausgeführten Konstructionen. Nauk.-Doslid. Inst. Mat. Meh. Harkiv. Univ. Geometričnil Zbirnik 2, 93–105 (1940). (Ukrainian. German summary) [MF 16947]

Es wird für die üblichen Zeicheninstrumente (Zirkel, Lineal, Winkel, Reisschiene) auf Grund der charakteristischen damit ausgeführten Operationen abstrakte Erklärungen gegeben, Postulate der "Konstruktibilität" der geometrischen Elemente aufgestellt. Lemoine'scher Symbol der Konstruktion $l_1R_1+l_2R_2+l_3R_3+l_4R_4$ wird durch zwei neue Symbole η und ξ ergänzt: wiederholte Versetzung der Zirkelenden und das wiederholte Anlegen des Lineals an den gegeben Punkt.

Author's summary.

✱Gambier, B. Cycles paratactiques. Mémor. Sci. Math., no. 104, Gauthier-Villars, Paris, 1944. 92 pp. (1 plate)

The term "paratactical circles" was first used by J. L. Coolidge [A Treatise on the Circle and the Sphere, Clarendon Press, Oxford, 1916]. The geometry of circles in space was brought up to date and given a unified synthetic presentation by J. Hadamard [Nouv. Ann. Math. (6) 2, 257–270, 289–320 (1927); Revista Mat. Hisp.-Amer. (2) 2, 81–87, 107–116, 137–143, 179–182 (1927)]. The book under review summarizes the progress made in this domain during the last two decades and presents a complete theory of parataxy. The new results are largely due to French writers, the author among them. The method of presentation differs considerably from Hadamard's, but is kept throughout on the collegiate level, with some few exceptions.

In the first quarter of the book the author deals with circles, and only later introduces directed circles (p. 22). The use of imaginaries is avoided, but the author pays tribute to their utility and power by devoting to them the last (fifth) chapter. For bibliographical references the author relies largely upon Hadamard's paper. A sheet of thirty very helpful figures is attached to the book.

N. A. Court (Norman, Okla.).

Thébault, Victor. Concerning pedal circles and spheres. Amer. Math. Monthly 53, 324–326 (1946). [MF 16724]

Turri, Tullio. La non necessità della ipotesi della continuità delle trasformazioni conservanti i gruppi armonici. Rend. Sem. Fac. Sci. Univ. Cagliari 13, 5–10 (1943). [MF 16222]

A biunique point transformation of the complex line which transforms two distinct points into two distinct points and transforms four points of any harmonic set into four points also harmonic is either a homology or an antihomology. In either case in the complex domain the hypothesis of continuity is not necessary.

V. G. Grove.

Turri, Tullio. Sulla non necessità della ipotesi della continuità delle trasformazioni conservanti i gruppi armonici. Rend. Sem. Fac. Sci. Univ. Cagliari 14, 53–54 (1944). [MF 16632]

Correction to the paper reviewed above.

Vişa, E. On the axiom of Pasch. Gaz. Mat., Bucureşti 51, 124–127 (1946). (Romanian) [MF 16557]

Ionescu-Bujor, C. On generalized polarities. Gaz. Mat., Bucureşti 51, 381–390 (1946). (Romanian) [MF 16559]

Hjelmslev, Johannes. Einleitung in die allgemeine Kongruenzlehre. III. Danske Vid. Selsk. Math.-Fys. Medd. 19, no. 12, 50 pp. (1942). [MF 15394]

[Parts I and II appeared in the same Medd. 8, no. 11; 10, no. 1 (1929).] The author discusses a plane geometry in which two points can always be connected by at least one straight line, but two different lines may have either a point or a whole segment in common. Congruence is defined by means of a group of transformations that is transitive both with respect to the points and to the straight lines. Special transformations, "reflections" that leave all the points of a straight line and no other points fixed, serve to define orthogonality.

An angle is called singular if the straight lines through its sides are different and have a whole segment in common. A segment is called singular if it lies on more than one straight line. The following generalization of the theorems of Saccheri-Lambert is proved. If there is a triangle with nonsingular sides and angles the sum of whose angles is, respectively, equal to, larger than, or smaller than two right angles, then the same holds true for every triangle. Furthermore, triangles with singular sides or angles are studied.

Two "neighbor points" lie on a singular segment. Two straight lines are called neighbors if each point on one of them has a neighbor point on the other. Neighborhood is transitive. Two intersecting straight lines are neighbors if and only if they have a segment in common. In the set of the neighbor points of a point an "almost-Euclidean" geometry holds: the sum of the angles of a triangle differs from two right angles only by a singular angle. If all the $E(s)$ are congruent, then this geometry is Euclidean. Here $E(s)$ denotes the intersection of all the straight lines through the singular segment s. By identifying neighbor points and neighbor straight lines, the author arrives at a "geometry in the large" which satisfies the axioms of projective geometry and the congruence axioms. If all the $E(s)$ are congruent and if coordinates (u, v) are introduced in the geometry in the large, then every point of the original plane can be described through dual coordinates $(u+\epsilon u', v+\epsilon v')$, $\epsilon^2=0$. In the general case, $E(s)$ contains the union of all the mul-

tiples of the singular segment s. The author studies extensively the case where $E(s)$ is equal to this union for every s.
P. Scherk (Saskatoon, Sask.).

Hjelmslev, Johannes. Einleitung in die allgemeine Kongruenzlehre. IV. Danske Vid. Selsk. Math.-Fys. Medd. 22, no. 6, 40 pp. (1945). [MF 15407]

Let R_n denote an n-space in which there always exists at least one straight line between any two points, but the intersection of two different straight lines may be a whole segment. In the R_n, a group of transformations ("motions") is given that satisfies certain conditions; in particular, it is transitive both with respect to the points and to the straight lines. Congruence and orthogonality are defined by means of this group [cf. the preceding review]. If each of m straight lines through a point is perpendicular to the others, they determine a subspace R_m that satisfies the original axioms. Each R_m determines one and only one "reflection," that is, an involutionary motion that leaves all the points of the R_m and no other points fixed ($0 \leqq m < n$).

The author studies products of reflections and the decomposition of motions into such products, and he extends classical theorems on such decompositions. Thus he proves, for example: any motion in the R_n can be expressed as the product of at most $n+1$ reflections at suitable $(n-1)$-spaces; any motion with the fixed point 0 can be represented as the product of $n-1$ reflections at straight lines through 0 multiplied by the identity or another such reflection if n is even or multiplied by the identity or a reflection at 0 if n is odd.
P. Scherk (Saskatoon, Sask.).

Rossier, Paul. Sur l'équation de Chasles. C. R. Séances Soc. Phys. Hist. Nat. Genève 62, 95–97 (1945). [MF 16532]

If the function $f(x, y)$ possesses partial derivatives of the first two orders and satisfies the relation

$$f(a, b)+f(b, c)=f(a, c),$$

then it must take the form $f(x, y)=\varphi(x)-\varphi(y)$. This theorem justifies the usual definition of non-Euclidean distance as the logarithm of a cross ratio. *H. S. M. Coxeter.*

Rosenfeld, B. A. Die innere Geometrie der Geradenmannigfaltigkeit des elliptischen Raumes. Uchenye Zapiski Moskov. Gos. Univ. Matematika 73, 49–58 (1944). (Russian. German summary) [MF 15194]

The introduction of the Pluecker coordinates for straight lines in 3-space leads, as is known, to the consideration in a projective 5-space of a 4-dimensional quadric P whose points correspond to these straight lines; this situation is studied in the present paper on the basis of an elliptic metric induced [as A. P. Norden has shown in an apparently unpublished work] by the metric (assumed to be elliptic) of the given 3-space. In terms of this metric, P is shown to be the locus of points equidistant from each of two planes (central planes), all the distances being equal to $\pi/4$; furthermore, P can be split in two ways into families of spheres, the centers of each family covering one of the central planes; such a sphere is the map of a set of Clifford parallels in the original 3-space. An important part in the study is played by the concept of a "quadratic," a locus of points whose distance from a given line is $\pi/4$; thus, in the 5-space the intersection of P with a 3-space containing one line in each of the central planes is a quadratic; P can be split in ∞^2 ways into ∞^2 quadratics; such a quadratic is the map of a congruence of all lines perpendicular to a given line in 3-space; on the other hand, quadratics in the original 3-space are mapped into circles of radius $\pi/4$ which are great circles on the above mentioned spheres on P. The geodesics on P are maps of helicoids of the 3-space. Generalization to higher dimensions is briefly mentioned. *G. Y. Rainich.*

Bruins, E. M. Generalization of some elementary theorems in n-ary Ω-geometry. Nederl. Akad. Wetensch., Proc. 48, 198–205 = Indagationes Math. 7, 3–10 (1945). (Dutch) [MF 15790]

Formulae for angles, distances and volumes in n-dimensional non-Euclidean geometry. The absolute quadratic form $(\Omega'x)^2$ and the set of points $(u'x_i)$ $(i=1, 2, \cdots)$ have the invariants $(\Omega'\Omega)^2$ and $\Omega_{ik}=(\Omega'x_i)(\Omega'x_k)$, by means of which the symmetrical invariants

$$\Delta(x_{i_1}, x_{i_2}, \cdots, x_{i_d})=\det |\Omega_{rs}|$$

and the absolute invariants

$$\frac{1}{(d-1)!^2}\frac{\Delta(x_{i_1}, \cdots, x_{i_d})}{\Omega_{i_1i_1}\cdots\Omega_{i_di_d}}=\sin^2 k_{d-1}T(x_{i_1}, \cdots, x_{i_d})$$

can be obtained; k_{d-1} is a constant and T is called the $(d-1)$-dimensional Ω-volume of the simplex $(x_{i_1}, \cdots, x_{i_d})$. The author considers the d distances of two skew spaces of $d-1$ and $n-d-1$ dimensions, respectively. If the vertices of a simplex are divided into two sets $x_1, \cdots, x_d$ and $x_{d+1}, \cdots, x_n$ forming the simplices B and G, the formula

$$\pm \sin V=\frac{(n-d-1)!(d-1)!}{(n-1)!}\sin B \sin G \prod_{i=1}^{d} \sin^2 k_i\delta_i$$

is given for the volume of the simplex, δ_i being the distances of the spaces containing B and G. Generalization of Stewart's theorem and of the cosine theorem in spherical trigonometry. *O. Bottema* (Delft).

De Baggis, Henry F. Hyperbolic geometry. I. A theory of order. Rep. Math. Colloquium (2) 7, 3–14 (1946). [MF 16129]

Poursuivant des recherches de MM. Menger et Jenks, l'auteur donne un système de six postulats impliquant les postulats de la liaison de Hilbert et ne faisant intervenir que les opérations de jonction et d'intersection. Définissant comme Jenks [Proc. Nat. Acad. Sci. U. S. A. 26, 277–279 (1940); ces Rev. 1, 261] la notion de point situé entre deux autres, il prouve que les postulats de l'ordre de Hilbert sont vérifiés. Les considérations de l'article sont applicables à la géométrie hyperbolique plane mais non à la géométrie euclidienne. L'indépendance des postulats est montrée par des exemples. Le système des postulats posés est simple; les déductions sont naturelles et harmonieuses. *C. Pauc.*

Menger, Karl. New projective definitions of the concepts of hyperbolic geometry. Rep. Math. Colloquium (2) 7, 20–28 (1946). [MF 16131]

Dans cette note sont proposées des définitions de la congruence, de la perpendicularité et du parallélisme ne faisant intervenir que les opérations de jonction et d'intersection (définitions projectives) en supposant que le système des postulats régissant celles-ci permette l'introduction de points idéaux, ce qui est par exemple le cas pour le système de De Baggis signalé plus haut [cf. théorème d'immersion de Winternitz, Ann. of Math. (2) 41, 365–390 (1940); ces Rev. 1, 260]. Un programme de recherches est posé, à savoir: compléter ce système de manière à ce que soient vérifiées les propriétés de congruence et de parallélisme du plan

hyperbolique. Aucune allusion n'est faite à la théorie moderne des espaces projectifs sur des corps algébriques généraux, en particulier sur des corps ordonnés; l'espace défini par le système de De Baggis est un ensemble "convexe" d'un tel espace. *C. Pauc* (Marseille).

Tomber, Marvin L. On perpendicularity in rational hyperbolic planes. Rep. Math. Colloquium (2) 7, 15–19 (1946). [MF 16130]

L'auteur étudie l'existence de perpendiculaires à une droite dans le "plan hyperbolique rationnel," c'est-à-dire le plan cayléen correspondant à une conique $Ax^2+2Bxy+Cy^2=D$, dont seuls les points intérieurs de coordonnées rationnelles sont admis. Voici un spécimen des théorèmes obtenus: si $A, C, D\neq 0$ sont rationnels, $\sqrt{(A/C)}$ et B irrationnels, B n'étant pas un irrationnel quadratique, il n'existe aucun couple de droites perpendiculaires. *C. Pauc* (Marseille).

Convex Domains, Integral Geometry

Haupt, Otto. Über Verallgemeinerungen des Böhmerschen und verwandter Ovalsätze. Abh. Math. Sem. Hansischen Univ. 15, 130–164 (1943). [MF 15834]

A classical theorem by P. Böhmer [Math. Ann. 60, 256–264 (1905)] states that, if the conic determined by any five sufficiently close points on an oval with sufficient differentiability is an ellipse, then the conic determined by any five points of the oval is an ellipse. Analogous results have been obtained for parabolically and hyperbolically curved arcs by Carleman [Vierteljschr. Naturforsch. Ges. Zürich 85 Beiblatt (Festschrift Rudolf Fueter), 61–63 (1940); these Rev. 2, 260], Mohrmann [Math. Ann. 72, 285–291 (1912)] and others. The author generalizes the above results by replacing the family of conics by a family of curves which he calls order characteristics and whose properties are explicitly postulated. The classical theorems then appear as special cases of theorems which are stated in topologically invariant form. In the classical theorems the parabolas appear in their usual role of separating the ellipses and the hyperbolas. In his discussion the author shows that the theorems are essentially unchanged if the parabolas are replaced by the hyperbolas with fixed eccentricity f, while the ellipses and hyperbolas are replaced by the conics with eccentricity less than f and greater than f, respectively. Extensive use is made by the author of his earlier work on relative ordering [Monatsh. Math. Phys. 40, 1–53 (1933)]. *S. B. Jackson* (College Park, Md.).

Santaló, L. A. Verallgemeinerung eines Satzes von T. Kubota über Eilinien. Tôhoku Math. J. 48, 64–67 (1941). [MF 16348]

The author establishes the following theorem. Let the oval K have a continuous radius of curvature whose extreme values are denoted by R_m and R_M. Let k be a second oval having continuous curvature and whose radius of curvature r satisfies the relation $R_m<r<R_M$. Then K and k can intersect in no more than two points. The proof is by the methods of integral geometry. The theorem is a direct generalization of one by Kubota [same J. 47, 96–98 (1940); these Rev. 2, 12]. The author notes that his theorem was obtained by different methods by H. Brunn in 1889 [Über Curven ohne Wendepunkte, Habilitationsschrift, München, 1889]. *S. B. Jackson* (College Park, Md.).

Santaló, L. A. Geometric probabilities and integral geometry. Bol. Fac. Ingen. Montevideo 3, (Año 10), 91–113 (1945). (Spanish) [MF 16166]

Nöbeling, Georg. Die Formel von Poincaré für beliebige Kontinuen. Abh. Math. Sem. Hansischen Univ. 15, 120–126 (1943). [MF 15832]

Let K and K' be two plane curves of finite or infinite lengths $L(K)$ and $L(K')$, respectively. Furthermore, let $D(K, K')\leqq\infty$ be the number of points common to K and K'. If K is held fast and K' is subjected to all possible rigid motions, $D(K, K')$ is a function of the position of K'. Poincaré has shown that

$$\int D(K, K')\dot{K}'=4L(K)L(K'),$$

where $\dot{K}'$ is the kinematic density and the integration is taken over all positions of K'. [See Blaschke, Vorlesungen über Integralgeometrie, vol. I, Hamburger Math. Einzelschr. 20, Teubner, Leipzig, 1936, pp. 23–24, for a modern proof.]

In this paper this result is extended to include the cases where K and K' are two arbitrary continua in the plane (that is, connected, closed and bounded sets which contain more than one point). The proof first establishes the formula when K and K' are locally connected and hence have finite lengths in the sense of Gillespie. If either of the continua, say K, is not locally connected, $L(K)=\infty$; the proof is then completed by showing that in this case the integral on the left of the formula is also infinite.

C. B. Allendoerfer (Haverford, Pa.).

Bol, G. Beweis einer Vermutung von H. Minkowski. Abh. Math. Sem. Hansischen Univ. 15, 37–56 (1943). [MF 15828]

For any convex body B in 3-space the inequality

$$O^2-3MV\geqq 0 \tag{1}$$

is satisfied. Here O is the surface area of B, V its volume and M its total mean curvature. The author proves a conjecture of Minkowski, according to which the equality sign in (1) holds only if B is a "Kappenkörper" of a sphere, that is, if B can be generated by adding to a sphere a finite or denumerable number of nonoverlapping tangential cones. The proof is based on an analysis of the family of "interior" parallel bodies $B(\lambda)$, which consist of the interior points of B having a distance not less than $-\lambda$ from the boundary of B. Denoting by $O(\lambda)$ the surface area of $B(\lambda)$, the author considers the expression M^*, where $2M^*$ is defined as the left-hand derivative of the function $O(\lambda)$ for $\lambda=0$. It can then be proved that

$$(2)\quad M^*\geqq M, \qquad (3)\quad O^2-3M^*V\geqq 0.$$

Thus the equality sign in (1) implies equality in (2) and (3). The author proves that the equality sign in (3) holds only for the "tangential bodies" of a sphere, that is, for the convex bodies bounded by the tangent planes of a sphere at the points of a closed set which do not all lie on a hemisphere. The special tangential bodies of a sphere for which (2) holds are then shown to be "Kappenkörper."

The same question is also solved for "relative geometry" in 3 dimensions, where O and M are replaced by more general mixed volumes. *F. John* (New York, N. Y.).

Bol, G. Einfache Isoperimetriebeweise für Kreis und Kugel. Abh. Math. Sem. Hansischen Univ. 15, 27–36 (1943). [MF 15827]

The author gives a very simple proof of the isoperimetric inequality

$$(1) \qquad \Delta = L^2 - 4\pi O \geqq 0$$

for polygons in the plane. (Here L denotes length; O, enclosed area.) The proof is based on an explicit representation of the "isoperimetric defect" Δ for "interior" parallel polygons. By passing to the limit one obtains (1) for arbitrary Jordan curves, though this derivation does not permit one to decide when the equality sign holds.

The isoperimetric inequality for convex bodies in space can be reduced to the two inequalities

$$(2) \qquad M^2 - 4\pi O \geqq 0,$$
$$(3) \qquad O^2 - 3MV \geqq 0$$

where V is the volume, M the total mean curvature and O the surface area. The author gives an elementary derivation of (3) for polyhedra, which, however, has to make use of (2). The methods of proof are similar to those used by the author in the paper reviewed above. *F. John.*

Bouffard, Jean. Le problème des isopérimètres. Revue Sci. (Rev. Rose Illus.) 82, 403–420 (1944). [MF 16544]

Expository article.

Soloviov, P. Sur une résolution du problème des isopérimètres. Nauk.-Doslid. Inst. Mat. Meh. Harkiv. Univ. Geometričnil Zbirnik 2, 135–143 (1940). (Ukrainian. French summary) [MF 16951]

Démonstration simple de la formule de L. A. Santaló [Abh. Math. Sem. Hamburgischen Univ. 11, 222–236 (1936)] pour le déficit isopérimétrique et la solution du problème des isopérimètres (plan) qui en découle. *Author's summary.*

Soloviov, P. Sur le barycentre de courbure. Nauk.-Doslid. Inst. Mat. Meh. Harkiv. Univ. Geometričnil Zbirnik 2, 115–133 (1940). (Ukrainian. French summary) [MF 16950]

Soit O une courbe plane convexe fermée possédant un nombre fini des sommets. Il existe une seule courbe C^0 de longueur algébrique nulle, parallèle à la courbe O. Le but principal de l'auteur est de démontrer: (1) la suite des evolventes $i$èmes de la courbe C^0 se resserre, sous la condition que la longueur algébrique de chacune est nulle, à un point limite G; (2) ce point limite G est le barycentre de courbure de la courbe donnée O; (3) la suite indiquée des evolventes $i$èmes de la courbe C^0 peut être choisie d'une manière unique. Plusieurs propriétés des courbes de longueur algébrique nulle sont données. *From the author's summary.*

Hirschwald, L. Des courbes à largeur constante et le problème de Buffon. Nauk.-Doslid. Inst. Mat. Meh. Harkiv. Univ. Geometričnil Zbirnik 2, 9 (1940). (Russian. French summary) [MF 16941]

This is an announcement of results. The summary is as follows. On sait que toutes les courbes à largeur constante h ont le même périmètre égal à πh et d'autre part que le périmètre de la courbe convexe avec la corde maximum égale à h est au plus égal à πh. Dans cette note je montre qu'on peut obtenir la démonstration de ces deux propriétés comme une conséquence immédiate de la résolution d'un problème du calcul des probabilités qui est une généralisation du problème célèbre de Buffon.

Blanc, Eugène. Sur une généralisation des domaines d'épaisseur constante. C. R. Acad. Sci. Paris 219, 662–663 (1944). [MF 15298]

Let $R>0$. An R-convex plane set is the intersection of closed circles of radius R. The R-complement $\bar{E}$ of a set E is the intersection of all the closed circles of radius R whose centers lie in E. Let E be R-convex. Then the R-complement of the R-complement of E is E. If $H(\varphi)$ and $\bar{H}(\varphi)$ are the gauge functions of the boundaries of E and $\bar{E}$, then $H(\varphi)+\bar{H}(\varphi+\pi)=R$. The author states a few corollaries of this formula. *P. Scherk* (Saskatoon, Sask.).

Fejes, L. Über einen extremalen Polyeder. Math. Naturwiss. Anz. Ungar. Akad. Wiss. 59, 476–479 (1940). (Hungarian. German summary) [MF 15568]

The author proves the following theorem. Let F be a convex closed smooth surface and let P_n be a polyhedron of n vertices which is inscribed in F and whose surface area is maximal. Let C be an arbitrary vertex of P_n and let $C_1, \cdots, C_m$ be the vertices connected with C by an edge. Consider the vectors v_i, $i=1, \cdots, m$. One vertex of v_i is C, v_i is equal to $\overline{C_iC_{i+1}}$ ($C_{m+1}=C_1$) and v_i is in the plane CC_iC_{i+1} and is perpendicular to C_iC_{i+1}. Then the vector $v_1+\cdots+v_m$ is parallel to the normal of the surface F at C. *P. Erdös* (Stanford University, Calif.).

Herglotz, G. Über die Steinersche Formel für Parallelflächen. Abh. Math. Sem. Hansischen Univ. 15, 165–177 (1943). [MF 15835]

The author takes the spherical n-space S_n consisting of the boundary of the unit sphere in Euclidean $(n+1)$-space, and considers a closed, orientable, regular surface F (that is, an $(n-1)$-subspace) in S_n and the polar surface F^* of F. He uses Steiner's formula for the volume enclosed between F and the parallel surface F_t at distance t, namely $D(t)=M_0\varphi_0(t)+\cdots+M_{n-1}\varphi_{n-1}(t)$, where $M_0, \cdots, M_{n-2}$ are the "mean curvatures" and M_{n-1} is the area of F, while $\varphi_\nu(t)=\int_0^t \cos^\nu t \sin^{n-\nu-1} t\, dt$. Let

$$c_\nu = \varphi_\nu(\pi/2) = \omega_{n+1}/(\omega_{\nu+1}\omega_{n-\nu}),$$

where ω_ν is the surface area of the unit sphere in Euclidean ν-space. Let a be a fixed point of S_n, and denote by θ and θ^* the distances from a in S_n of a point of F and the corresponding point of F^*, respectively. Write

$$C_a{}^*\omega_n = \int_{F^*} \frac{\cos\theta d\sigma^*}{\sin^n \theta^*}, \quad V_a = -\int_F \frac{\varphi_0(\theta)\cos\theta^* d\sigma}{\sin^n\theta},$$

$$V_a{}^* = \int_{F^*} \frac{\{c_0-\varphi_0(\theta^*)\}\cos\theta d\sigma^*}{\sin^n\theta^*},$$

where $d\sigma$ and $d\sigma^*$ are the elements of area on F and F^*. The author proves that, if $0\leqq\theta<\pi/2$ on F and $0<\theta^*<\pi$ on F^*, we have

$$c_0M_0+c_2M_2+\cdots = \tfrac{1}{2}C_a{}^*\omega_{n+1} - \tfrac{1}{2}V_a\{1+(-1)^n\},$$
$$c_1M_1+c_3M_3+\cdots = -\tfrac{1}{2}V_a\{1-(-1)^n\} - V_a{}^*.$$

It is noted that $C_a{}^*$ can be interpreted as the Kronecker index of F^* about a. Closely analogous formulae are developed for hyperbolic n-space. *H. P. Mulholland* (Beirut).

Hadwiger, H. Eine Erweiterung des Steiner-Minkowskischen Satzes für Polyeder. Experientia 2, 70 (1946).

In n-dimensional Euclidean space let A be a polyhedron

and K_r a sphere of radius r and center $(x_1, \cdots, x_n)$. Let $\varphi(AK_r)$ be the Euler characteristic of the intersection of A and K_r; φ is defined to be zero when AK_r is empty. The author states without proof the theorem that

$$J(A,r)=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\varphi(AK_r)dx_1\cdots dx_n$$

is a polynomial in r of degree not more than n. In case A is convex, this reduces to a theorem of Steiner and Minkowski. In that case φ is either 0 or 1, depending on whether AK_r is empty or not, and J reduces to the volume of the outer parallel body to A at distance r, which is then a polynomial of degree less than or equal to n. *J. W. Green.*

Geppert, Harald. Sopra una caratterizzazione della sfera. Ann. Mat. Pura Appl. (4) **20**, 59–66 (1941). [MF 16597]

W. Scherrer has demonstrated that the total torsion $T=\oint\tau(s)ds$ (s, arc length; $\tau(s)$, torsion) vanishes for all the closed curves on a surface S if and only if S is a sphere or a plane [Vierteljschr. Naturforsch. Ges. Zürich **85** Beiblatt (Festschrift Rudolf Fueter), 40–46 (1940); these Rev. **3**, 89]. This paper presents two simpler proofs of this theorem. The first is based on the fact that T and $T_g=\oint\tau_g(s)ds$ ($\tau_g(s)$, geodesic torsion) differ by a multiple of 2π. In the second proof, the author studies the neighborhood of a particular point O with the help of a coordinate system in which two axes have the principal directions at O and the third has the direction of the normal to the surface at O. Then $\tau(s)$ is calculated directly for a specially chosen family of curves and the vanishing of T for this family leads to the desired conclusion by elementary reasoning.

A. Schwartz (State College, Pa.).

Signorini, A. Sopra una caratterizzazione della sfera. Ann. Mat. Pura Appl. (4) **20**, 211–212 (1941). [MF 16602]

H. Geppert has furnished two proofs of the fact that the total torsion vanishes for all the closed curves on a surface S if and only if S is a sphere or plane [see the preceding review]. In this paper the first proof is simplified for real surfaces by using the following facts: (1) on any surface the curves along which the geodesic torsion vanishes are the lines of curvature; (2) planes and spheres are characterized by the fact that all their lines are lines of curvature.

A. Schwartz (State College, Pa.).

Algebraic Geometry

***van der Waerden, B. L. Einführung in die algebraische Geometrie.** Dover Publications, New York, N. Y., 1945. vii+247 pp. $3.50.

Photographic reprint. The original was published by Springer, Berlin, 1939.

Severi, Francesco. I fondamenti della geometria numerativa. Ann. Mat. Pura Appl. (4) **19**, 153–242 (1940). [MF 15085]

The author is concerned with showing that the calculus of conditions of enumerative geometry can be put on a rigorous footing by using the theory of the base for equivalence (in the arithmetic sense) previously developed by him [R. Accad. Italia. Mem. Cl. Sci. Fis. Mat. Nat. **5**, 239–283 (1934)]. The difficulties of the enumerative theory which arise from the existence, in many cases, of irrelevant "degenerate" solutions of an enumerative problem are overcome by what amounts, essentially, to the application of the theory of the base to open varieties, that is, varieties of the form $V-E$, where E is a subvariety of V. The author establishes the existence of such a base for arithmetic equivalence and by means of this is able to prove the existence of a finite number of characteristics of each relevant dimension for a given irreducible system of algebraic varieties. The method is illustrated by a detailed discussion of the characteristic problem for systems of conics in a plane.

J. A. Todd (Cambridge, England).

Severi, Francesco. Über die Darstellung algebraischer Mannigfaltigkeiten als Durchschnitte von Formen. Abh. Math. Sem. Hansischen Univ. **15**, 97–119 (1943). [MF 15831]

Given a system R_0: $f_i(x_0, x_1, \cdots, x_n)=0$ of h homogeneous equations, the Kronecker method of indeterminates and the successive elimination of $x_n, x_{n-1}, \cdots, x_2$ lead to resultant systems $R_1, R_2, \cdots, R_{n-1}$. If the reference frame of the coordinate system is in general position, then each irreducible common factor $g(x_0, x_1, \cdots, x_{n-i})$ of the forms of the system R_i determines uniquely a solution ξ of R_0, of dimension $k=n-i-1$ (possibly an embedded solution, that is, one which is a specialization of a solution of higher dimension). The author proposes to attach to ξ the multiplicity ρ if g^ρ is the highest power of g that divides the forms in R_i. He proposes a second definition based on the method of specialization in the complex domain: if $h=n$ and if $F_i(x)=0$, $i=1, 2, \cdots, n$, is an approximating system of equations to the original one having exactly $d_1d_2\cdots d_n$ distinct solutions (d_i, the degree of f_i), then the multiplicity of a zero-dimensional solution ξ of R_0 is the least number of solutions of the approximating system (over all such systems) which approach ξ. The author does not touch on the crucial question of whether and how his definitions of multiplicity of an embedded solution or of a solution of the "wrong" dimension can be incorporated in the general intersection theory [see in this connection O. Perron's criticism of the present paper [Math. Z. **49**, 654–680 (1944); these Rev. **6**, 185]].

The rest of the paper deals with the concept of "complete simple intersection." If ξ is the only nonembedded solution and if its multiplicity is 1, then the corresponding k-dimensional irreducible variety V (having ξ as general point) is said to be the complete intersection of the h given hypersurfaces f_i. If, moreover, there are no embedded solutions, then V is called the complete simple intersection. The following theorem is proved. If a space curve C, free from singularities, is the complete simple intersection of the h surfaces f_i and if $h\leqq 4$, then the ideal $A=(f_1, f_2, \cdots, f_h)$ contains every form of a sufficiently high degree which vanishes on C. The proof is based on properties of linear systems and on the postulation formula for C. In view of the very restrictive hypotheses under which the author proves his theorem ($h\leqq 4$, $n=3$, $k=1$) and of the involved nature of the proof, the following observation may be made. The author tacitly assumes that his definitions imply the following property of a complete simple intersection V: at every simple point P of V, $n-k$ of the hypersurfaces f_i have at P linearly independent tangent hyperplanes. But that means that the ideal $\mathfrak{A}$ coincides with the prime H-ideal $\mathfrak{p}$ of V locally, at each simple point of V. If V is free from singularities, then it follows that $\mathfrak{A}$ can differ from $\mathfrak{p}$ only by an irrelevant primary component, and this is exactly the desired theorem, without any restrictions. *O. Zariski.*

Severi, Francesco. Il teorema di Riemann-Roch sopra le superficie per curve dotate di componenti multiple. Boll. Un. Mat. Ital. (2) 4, 1–7 (1942). [MF 16048]

The author remarks that the usual proof of the Riemann-Roch theorem concerning the dimension of the complete linear system $|C|$ defined by an algebraic curve C lying on an algebraic surface is incomplete as it stands if the curve C is reducible and possesses multiple components. The note is devoted to repairing the deficiency thus pointed out.

J. A. Todd (Cambridge, England).

Severi, Francesco. Sulle sezioni spaziali delle varietà algebriche normali. Boll. Un. Mat. Ital. (2) 4, 81–82 (1942). [MF 16057]

The linear sections of an algebraic normal variety of superficial irregularity zero (in particular, of any rational normal algebraic variety) are all normal. *J. A. Todd.*

Severi, Francesco. Sul teorema fondamentale dei sistemi continui di curve sopra una superficie algebrica. Ann. Mat. Pura Appl. (4) 23, 149–181 (1944). [MF 16614]

The author comes back once more to the long discussed problem of the completeness of the characteristic series of a complete continuous system of curves on an algebraic surface F. The algebro-geometric proof proposed by B. Segre [same Ann. (4) 17, 107–126 (1938)] is critically examined and pronounced incomplete. The author constructively analyzes the difficulties which, in his opinion, would have to be overcome in order to render Segre's proof completely rigorous. The positive contribution of the present paper is a proof that from the completeness of the characteristic series on a regular curve (established by Poincaré by transcendental methods) follows by purely algebro-geometric considerations the completeness of the characteristic series on any curve C on which the canonical system of F cuts out a complete series. *O. Zariski.*

Okugawa, Kôtaro. Remarks on O. Zariski's paper. Mem. Coll. Sci. Kyoto Imp. Univ. Ser. A. 23, 437–444 (1941). [MF 15818]

Given an irreducible r-dimensional algebraic variety V over a ground field K of characteristic zero, with general point $(\xi_1, \xi_2, \cdots, \xi_n)$, and given an irreducible s-dimensional subvariety V_s of V, let $\mathfrak{p}_s$ be the prime ideal of V_s in the coordinate ring $\mathfrak{o}=K[\xi]$ and let $\eta_1, \eta_2, \cdots, \eta_r$ be elements of $\mathfrak{o}$ such that (1) the ξ's are integral over $K[\eta]$, (2) $\eta_1, \eta_2, \cdots, \eta_s$ are algebraically independent mod $\mathfrak{p}_s$ (over K). One then has congruences of the form $f_i(\eta_1, \eta_2, \cdots, \eta_s, \eta_{s+i}) \equiv 0 \pmod{\mathfrak{p}_s}$, $i=1, 2, \cdots, r-s$, where the f_i are irreducible polynomials over K. The following theorem was proved by the reviewer, first in the case of an algebraically closed field K [Amer. J. Math. 61, 249–294 (1939)], and then in the general case, which he reduced to the former case by the method of algebraic ground field extensions [Amer. J. Math. 62, 187–221 (1940); these Rev. 1, 102]. A necessary and sufficient condition that V_s be simple for V and that $f_1, f_2, \cdots, f_{r-s}$ be uniformizing parameters of V_s is that there should exist an element ω in $\mathfrak{o}$ such that $G_\omega'(\eta_1, \eta_2, \cdots, \eta_r, \omega) \not\equiv 0 \pmod{\mathfrak{p}_s}$, where $G(\eta_1, \eta_2, \cdots, \eta_r, z)$ is the norm of $z-\omega$ with respect to the algebraic extension $K(\xi)/K(\eta_1, \eta_2, \cdots, \eta_r)$. If K^* is an algebraic extension of the ground field K, the reviewer's procedure consisted in passing from the integral domain $\mathfrak{o}$ to a suitably defined integral domain $K^*\cdot\mathfrak{o}$ and in analyzing the manner in which $\mathfrak{p}_s$ splits in $K^*\cdot\mathfrak{o}$. In the present paper the original proof is carried out according to a slightly different procedure. Let $R=K[x_1, x_2, \cdots, x_n]$, let $\mathfrak{p}$ be the prime ideal of V in R and let $R^*=K^*[x_1, x_2, \cdots, x_n]$. Instead of first passing to the residue class ring $\mathfrak{o}=R/\mathfrak{p}$ and then extending $\mathfrak{o}$ to the integral domain $K^*\cdot\mathfrak{o}$, the author first extends the prime ideal $\mathfrak{p}$ to the polynomial ring R^* and then operates in the residue class ring $\mathfrak{o}^*=R^*/R^*\mathfrak{p}$ (which may have zero divisors). Such concepts as intervene in the original proof (quotient rings, local uniformizing parameters, formal power series extensions, etc.) can be defined also in the zero divisor ring $\mathfrak{o}^*$. *O. Zariski.*

Zappa, Guido. Sugli ipergruppi di corrispondenze ad indici limitati sopra una curva algebrica. Ann. Mat. Pura Appl. (4) 20, 291–312 (1941). [MF 16607]

A set G of algebraic correspondences on an irreducible algebraic curve C is called a multigroup or hypergroup if the product of any two correspondences in G is the sum of irreducible correspondences which are again elements of G. Hypergroups of correspondences were first introduced and studied by G. Ascoli [same Ann. (4) 6, 85–112 (1929)]. For an arbitrary integer $r\geqq 2$ and for an arbitrary set of $r-1$ elements S_i in G there is defined in an obvious fashion a curve $\{S_1, \cdots, S_{r-1}\}$ on the r-fold direct product V_r of C. If Γ is any curve on V_r, the indices of Γ are defined as the intersection numbers of Γ with $P_1\times V_{r-1}$ (P_1 fixed) and of Γ with $V_{r-1}\times P_r$ (P_r fixed). If the indices of any irreducible component of the variable curve $\{S_1, \cdots, S_{r-1}\}$ are bounded the multigroup G is said to be properly bounded. If the indices of the irreducible correspondences in G are bounded then G is said to be improperly bounded. The main result is the derivation of the most general type of properly bounded multigroups of correspondences. Let W be an algebraic curve in $(n, 1)$ correspondence ω with C such that the involution of order n on W which is defined by the correspondence is generated by a group H_0 of birational transformations of W into itself (in other terms, the field of rational functions on W is a normal extension of the field of rational functions of C). Let, moreover, H be a group of birational transformations of W into itself such that $H\supseteq H_0$. Then the correspondences $\omega^{-1}\tau\omega$, $\tau\varepsilon H$, form a properly bounded multigroup and all such multigroups can be obtained in this fashion. A special study is made of improperly bounded multigroups of correspondences (3, 3).

O. Zariski (Urbana, Ill.).

Châtelet, François. Les correspondances birationnelles à coefficients rationnels sur une courbe. C. R. Acad. Sci. Paris 222, 351–353 (1946). [MF 16007]

If γ is an algebraic curve of genus p belonging to the rational field R, that is, represented by one or more equations with rational coefficients, the birational group of γ in R consists of all the birational transformations of γ in itself which are representable by equations with rational coefficients. This group is finite if $p>1$, and then its transformations can be obtained by means of algebraic operations. The author has already studied the case $p=0$ [Ann. Sci. École Norm. Sup. (3) 61, 249–300 (1944); these Rev. 7, 323]. Now he investigates how the group above can be determined for $p=1$ by transforming γ in a plane cubic $C: y^2=x^3-ax-b$ (a, b rational, $4a^3-27b^2\neq 0$) by means of a suitable birational transformation with algebraic coefficients. If γ is neither harmonic nor equianharmonic, and so $ab\neq 0$, the birational group of γ in R is isomorphic either with the group of the rational points of C or with the birational group of C in R. The second alternative occurs if C and γ are of the same class, but the converse is not true.

B. Segre (Manchester).

Châtelet, François. Les êtres géométriques d'un corps abstrait. Ann. Univ. Lyon. Sect. A. (3) **8**, 5–28 (1945). [MF 16288]

A sketch, from the ideal-theoretic point of view, of the basic notions required in algebraic geometry over a ground field which is not assumed to be algebraically closed.

D. B. Scott (London).

Abellanas, Pedro. Normal algebraic surfaces over a perfect coefficient field of arbitrary characteristic. Revista Mat. Hisp.-Amer. (4) **5**, 221–230 (1945). (Spanish) [MF 15883]

The representation of the neighbourhood of a simple algebraic branch of an algebraic surface over the field of complex numbers, due to Hensel [Acta Math. **23**, 339–416 (1900)], is formally extended to the case of an arbitrary perfect base field of finite characteristic. It is thereby shown that the sufficient conditions given by Zariski [Amer. J. Math. **61**, 249–294 (1939)] for a subvariety of an algebraic variety to be simple are, in this case, also necessary. *D. B. Scott.*

Kasner, Edward. The conformal satellite of a general algebraic curve. Revista Unión Mat. Argentina **11**, 77–83 (1946). (Spanish) [MF 15671]

This is essentially the same as a paper already reviewed [Proc. Nat. Acad. Sci. U. S. A. **31**, 250–252 (1945); these Rev. **7**, 169]. *J. A. Todd* (Cambridge, England).

Meynieux, Robert. Sur une propriété caractéristique des courbes et surfaces algébriques. C. R. Acad. Sci. Paris **222**, 715–716 (1946). [MF 16173]

A new set of conditions is stated for a "converse" of the property that the locus of the mean centres of the n points in which a plane algebraic curve of order n is met by the lines in a fixed nonasymptotic direction is a straight line. The result can be extended to twisted curves and surfaces.

D. B. Scott (London).

Drach, Jules. Sur la théorie générale des courbes algébriques. C. R. Acad. Sci. Paris **222**, 117–120 (1946). [MF 15982]

✶Bos, W. J. Projectieve Differentiaalmeetkunde der Analytische Regeloppervlakken in R_4. [Projective Differential Geometry of the Ruled Surfaces in R_4]. Thesis, University of Amsterdam, 1942. x+83 pp. (Dutch)

Zusammenfassende Darstellung des Inhaltes von dreizehn Noten (die ersten fünf von R. Weitzenböck, die nächsten drei von Weitzenböck und Bos, die übrigen von Bos) in Nederl. Akad. Wetensch., Proc. **43**, 440–448, 548–556, 668–673, 797–804, 805–814 (1940); **44**, 1052–1057, 1185–1189 (1941); **45**, 17–19, 184–188, 350–353, 465–470, 540–545, 669–674 (1942) [siehe diese Rev. **2**, 17, 159, 160; **6**, 99, 104, 105 und die zwei folgenden Referate]. Ausserdem enthält die vorliegenden Dissertation noch ein wesentliches System von Differentialinvarianten, das einfacher ist als das in der fünften Mitteilung von Weitzenböck angegebene. Übrigens sei auf die Besprechungen der zitierten Noten verwiesen.

H. Freudenthal (Amsterdam).

Weitzenböck, R., und Bos, W. J. Zur projektiven Differentialgeometrie der Regelflächen im R_4. VI. Nederl. Akad. Wetensch., Proc. **44**, 1052–1057 (1941). [MF 15765]

[Cf. the preceding review.] Parts VI and VII of this series deal with the algebraic ruled surfaces in four dimensions of order 2 and 3, thus illustrating the general theory of earlier parts. Those of order 2 only require three dimensions at most and are conics, quadric cones and plane pencils of lines.

The cubic ruled surface is generated by a line $p+ta+\frac{1}{2}t^2b+\frac{1}{6}t^3q$, where t is a numerical parameter and p, a, b, q are four tensors each with ten components ($p=[p_{ij}]$, $i, j=1, 2, 3, 4, 5$, $p_{ij}=-p_{ji}$) of which p and q denote lines. The general cubic surface of this type is generated by a line which cuts a fixed line H and fixed conic K at related points. It meets H at the pivotal point (Heftpunkt), and H is the locus of the pivotal point as the generator varies. (All ruled surfaces in [4] possess such a locus of such points, but usually it is a curved locus.)

Properties are worked out. The surface has a cubic equation in line coordinates, also in prime coordinates. It contains ∞^2 conics, each meeting each generator once. Two arbitrary points on the surface determine one generator each, and one such conic which passes through them. Four lines with a single line transversal determine one surface containing all five. *H. W. Turnbull* (St. Andrews).

Weitzenböck, R., und Bos, W. J. Zur projektiven Differentialgeometrie der Regelflächen im R_4. VII. Nederl. Akad. Wetensch., Proc. **44**, 1185–1189 (1941). [MF 15769]

This continues the investigation initiated in part VI [see the preceding review] by classifying such cubic ruled surfaces into skew and into developable surfaces, there being two kinds of the former and four kinds of the latter.

H. W. Turnbull (St. Andrews).

Weitzenböck, R. Ueber die Figur dreier Ebenen im R_5. Nederl. Akad. Wetensch., Proc. **44**, 907–913 (1941). [MF 15758]

Given three general planes in five dimensions, no two having a point in common, they possess ∞^2 line transversals, each line meeting ∞^1 planes of which these three are particular members. A twisted cubic solid M_3^3 is generated by these planes or alternatively by these lines. The given planes and their derived projective properties can then illustrate the invariant properties of the binary cubic. A particular pair of further planes chosen from the ∞^1 system gives the Hessian. The linear plane-complex $\lambda_1E_1+\lambda_2E_2+\lambda_3E_3$, where the E_i denote the three given planes, has double planes when the constants λ_i are suitably chosen. Such planes are in fact the generating planes of M_3^3. [A sequel to this paper has already been reviewed [same Proc. **45**, 215–216 (1942); these Rev. **6**, 99]. Cf. E. A. Weiss, Punktreihengeometrie, Teubner, Leipzig-Berlin, 1939, pp. 45 ff.; and, for the case of three $[k-1]$'s in $[2k-1]$, Turnbull, Philos. Trans. Roy. Soc. London. Ser. A. **239**, 233–267 (1942); these Rev. **3**, 304.] *H. W. Turnbull.*

Bottema, O. A (2, 2) congruence which contains a doubly infinite system of quartic reguli. Nederl. Akad. Wetensch., Proc. **49**, 72–74 = Indagationes Math. **8**, 36–38 (1946). (Dutch) [MF 16565]

If a (2, 2) congruence of lines contains ∞^2 reguli, each line belongs to ∞^1 of the reguli, two of which are degenerate. Baldus [Über Strahlensysteme, welche unendlich viele Regelflächen 2. Grades enthalten, dissertation, Erlangen, 1910] distinguished two cases, according as the degenerate reguli are cones or conics, but seems to have overlooked the possibility of both kinds occurring at once. The author shows that such a self-dual congruence consists of those tangents of a given quadric Q which meet a given tangent l.

The two degenerate reguli containing a given line g of the congruence are the section of Q by the plane gl, and the enveloping cone to Q from the point $(g\cdot l)$.

H. S. M. Coxeter (Toronto, Ont.).

Amin, A. Y. On the parametric representation of the surface of intersection of two quadric primals in four dimensions. Proc. Math. Phys. Soc. Egypt 2, no. 3, 3–6 (1944). [MF 16335]

The two quadric primals, which are in general position in four dimensions, are taken to be

$$ax^2+by^2+cz^2+dt^2+e\theta^2=0,\quad x^2+y^2+z^2+t^2+\theta^2=0$$

in terms of homogeneous coordinates referred to their common self-conjugate pentahedron. The quadrics have a surface of intersection through which five quadric cones pass, whose five vertices form this pentahedron. By considering the generating planes of three of these cones the author obtains equations which lead to a parametric representation of the typical point on the surface of intersection. The five homogeneous coordinates x, y, z, t, θ are then given by five cubic polynomials in homogeneous parameters $\xi:\eta:\zeta$, the coefficients of the polynomials being functions of the square roots of the differences of the coefficients a, b, c, d, e.

H. W. Turnbull (St. Andrews).

Alguneid, A. R. On the quadrinodal cubic surface, its harmonic and equianharmonic envelopes and its Milne envelopes. Proc. Math. Phys. Soc. Egypt 2, no. 2, 7–14 (1944). [MF 16337]

The planes which cut a cubic surface (in three dimensions) in harmonic cubic curves envelop a surface T of class six; those which cut it in equianharmonic cubic curves envelop a surface S of class four. These were introduced by W. P. Milne and then studied by him [J. London Math. Soc. 1, 7–12 (1926); Proc. London Math. Soc. (2) 26, 377–394 (1927)] and by A. L. Dixon [Proc. London Math. Soc. (2) 26, 351–362 (1927)]. The case when the cubic has three nodes was discussed by J. Blakey [Proc. Edinburgh Math. Soc. (2) 2, 168–180 (1931)]. The present author discusses the case of four nodes. The properties of a quadrinodal cubic surface are worked out, with particular reference to the above envelopes S and T. The sextic T degenerates into a quadric envelope and a residual quartic possessing the same four nodes and the same four tangent cones at the nodes as the cubic surface. The same property holds for S.

H. W. Turnbull (St. Andrews).

Edge, W. L. Conics on a Maschke surface. Proc. Edinburgh Math. Soc. (2) 7, 153–161 (1946). [MF 16211]

This is a sequel to the author's paper on Maschke quartic surfaces [same Proc. (2) 7, 93–103 (1945); these Rev. 7, 71]. These surfaces have equations

$$\begin{aligned}
\phi_1&\equiv\sum x^4-6(y^2z^2+x^2t^2+z^2x^2+y^2t^2+x^2y^2+z^2t^2),\\
\phi_2&\equiv\sum x^4+6(-y^2z^2-x^2t^2+z^2x^2+y^2t^2+x^2y^2+z^2t^2),\\
\phi_3&\equiv\sum x^4+6(y^2z^2+x^2t^2-z^2x^2-y^2t^2+x^2y^2+z^2t^2),\\
\phi_4&\equiv\sum x^4+6(y^2z^2+x^2t^2+z^2x^2+y^2t^2-x^2y^2-z^2t^2),\\
\phi_5&\equiv-2\sum x^4-24xyzt,\quad \phi_6\equiv-2\sum x^4+24xyzt
\end{aligned}$$

and they are closely associated with Klein's configuration of desmic tetrahedra derived from six linear complexes mutually in involution. Each of the fifteen differences $\phi_i-\phi_j$ gives one such tetrahedron of four planes. A line d passes through three vertices, one from each of three desmic tetrahedra. Each of eight planes through d touches M, any one of the six surfaces, at four distinct points, and meets M in two conics through these points. The same eight planes through d belong to two of the six surfaces. One surface has in all 240 such plane sections; 160 others meet the surface in pairs of conics having double contact.

Various properties are worked out, including those of the Hessian H of M. The Hessian contains 40 conics which form the complete intersection of H with the fundamental quadrics of Klein.

H. W. Turnbull (St. Andrews).

Emch, Arnold. New polylateral configurations on plane cubics and certain sextics in the projective plane. Tôhoku Math. J. 48, 25–33 (1941). [MF 16343]

The Cremona transformation $\rho x_i'=1/x_i$, $i=1, 2, 3$, leaves four points $(\pm1, \pm1, \pm1)$ latent, and also two cubic curves

$$(1)\qquad a_1x_1(x_2^2-x_3^2)+a_2x_2(x_3^2-x_1^2)+a_3x_3(x_1^2-x_2^2)=0,$$

$$(2)\qquad \sum_{1,2,3} a_1x_1(x_2^2+x_3^2)+2a_4x_1x_2x_3=0,$$

where the x_i are homogeneous coordinates referred to a triangle of reference. Call the vertices of the triangle A and the four latent points B. The four cofactors of the coefficients a_i in (2) may be taken as coordinates v_i ($i=1, 2, 3, 4$) of a point in three dimensions which perforce must lie on the surface

$$\Gamma_3:\ 2v_1v_2v_3-(v_1^2+v_2^2+v_3^2)v_4+v_4^3=0,$$

Cayley's quadrinodal cubic surface. This maps the plane of x_i on the surface of v_i; in particular, the four points B are mapped on the four nodes. [Cf. Emch, Amer. J. Math. 48, 21–44 (1926).] The author studies the cubics (2) and derives new properties of closed polygons, or systems of polygons, inscribed in a plane cubic curve, thus generalizing on classical results of Steiner. The proofs depend on the use of the above mapping.

The cubics (2) contain sets of twelve nodal cubics whose nodes form a configuration associated with two given points P and Q. A pencil of cubics (2) goes through two such points, and gives one such nodal set. All cubics (2) contain the three points A; two members of a pencil also have six points, called a Q_6, in common. This Q_6 consists of the six vertices of a complete quadrilateral which also are three pairs of corresponding points in the Cremona transformation T. Each such pair is a Steinerian couple on the cubic curve (tangents at which intersect on the curve). A set of four such Q_6 exist, each pair having such a couple in common. There are ∞^1 such sets. Certain closure properties are worked out for these and for their maps on Γ_3.

The analogue of Q_6 in three dimensions is a pentalateral Π_5 consisting of ten lines in which five arbitrary planes intersect. The cubic Γ_3 is one of ∞^5 such quadrinodal cubics which contain these ten lines. Use of a quartic surface containing the six edges of the tetrahedron of reference leads to a sextic curve of intersection with Γ_3, connected with a Π_5. Poristic properties emerge, and lead back to plane properties for the initial figure, concerning plane sextic curves of genus 7 passing through B and with double points at A.

H. W. Turnbull (St. Andrews).

Huff, G. B. Inequalities connecting solutions of Cremona's equations. Bull. Amer. Math. Soc. 52, 287–291 (1946). [MF 16192]

The character $(x)=(x_0; x_1, \cdots, x_\rho)$ of a complete and regular linear series of plane curves is determined by its order x_0 and the multiplicities x_i at the base points. The dimension d and genus p of the series are given by $x_0^2-\sum x_i^2=(xx)=d+p-1$ and $3x_0-\sum x_i=(lx)=d-p+1$.

An arbitrary character (x) is "proper" if there is a linear system having this character. If (x) is proper then for any proper (f) with $p=d=0$ we have $(fx)\geqq 0$. The author extends this result by showing that, for any (x) with $x_0>0$, $p\geqq 0$, $d\geqq 0$, $p+d>0$, and any proper (f) with $p=d=0$ and $f_0\geqq x_0$, we have $(fx)\geqq 0$. This theorem is used to give a simple proof of an earlier result of the author that any characteristic (x) with $x_0>0$, $p=0$, $d=2$, and such that $(fx)\geqq 0$ for all proper (f) with $p=d=0$ and $f_0<x_0$, is the characteristic of a homaloidal net. *R. J. Walker.*

Lesieur, Léonce. Deux effets de la rationalité de l'intersection de p hyperquadriques d'un espace E_n à n dimensions. C. R. Acad. Sci. Paris **220**, 808–810 (1945). [MF 15816]

The author draws attention to certain birational and rational transformations which arise by considering birational or rational correspondences between the variety common to p quadrics in n dimensions and a space of $n-p$ dimensions, when n is sufficiently large. *J. A. Todd.*

Gherardelli, Giuseppe. Un'osservazione sul modello minimo della varietà degli elementi punto-iperpiano incidenti di S_r. Boll. Un. Mat. Ital. (2) **4**, 83–84 (1942). [MF 16058]

The variety in question has been shown by Martinelli to be the prime section of the V_{2r} representing the pairs of points of two S_r's. The author gives a very simple direct proof that this variety is normal. *J. A. Todd.*

Dedò, M. Espressione analitica di alcuni gruppi di proiettività caratterizzati in modo differenziale. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **77**, 399–406 (1942). [MF 16256]

Explicit determination of the group of projectivities in a plane leaving invariant a differential element consisting of a point O and $k(\leqq 6)$ consecutive points following O on a general linear branch. For $k=6$ the group is cyclic of period 3, for $k<6$ it is a continuous group. The author shows that by means of a suitable transformation of coordinates the operations π_k of the group can be expressed in the form $\pi_k=\tau_k^{\,e}\pi_{k+1}$, where π_6 is a power of the projectivity $x'=\epsilon x$, $y'=\epsilon^2 y$ (ϵ a primitive cube root of unity) and the projectivities $\tau_5^{\,a}$, $\tau_4^{\,b}$, $\tau_3^{\,c}$, $\tau_2^{\,d}$, $\tau_1^{\,e}$, $\tau_0^{\,f}$ are respectively $x':y':1=x+ay:y:2ax+a^2y+1$; $x'=bx$, $y'=b^2y$; $x':y':1=x:y:cy+1$; $x':y':1=x:y:dx+1$; $x'=x-cy$, $y'=y$; $x'=x$, $y'=y+fx$. *J. A. Todd* (Cambridge, England).

Vaccaro, Giuseppe.. Sopra alcune correlazioni nulle dello spazio. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **78**, 60–67 (1943). [MF 16241]

Lage Sundet, Knut. On a construction for rational curves of class 3. Norsk Mat. Tidsskr. **28**, 17–19 (1946). (Norwegian) [MF 16692]

d'Orgeval, Bernard. Remarque sur la décomposition des courbes de diramation des plans multiples. C. R. Acad. Sci. Paris **222**, 320–322 (1946). [MF 16006]

Bachiller, Tomás Rodriguez. Sulle superficie del quarto ordine contenenti una conica. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **3**, 556–562 (1942). [MF 16870]

Differential Geometry

Werenskiold, W. A class of equal area map projections. Skr. Norske Vid. Akad. Oslo. I. **1944**, no. 11, 18 pp. (6 plates) (1945). [MF 16447]

The general class of equal area world map projections having parallel straight line parallels is considered. A number of well-established types obtained by simple choices of meridian curves are reviewed. Three new projections are developed which have polar lines of half-equatorial lengths and interrupted parabolic, sinusoidal, and elliptical curves for meridians. For each of these, designated, respectively, as the truncated parabolic, sinusoidal, and elliptical projections, tables are given for the coordinates of the projection and for the maximum angular distortion. The mean maximum angular distortion is calculated for each projection, showing that the mean distortion is comparable to other established equal area projections. *N. A. Hall.*

Maeda, Jusaku. A characteristic property of space curves of constant first affine curvature. Tôhoku Math. J. **48**, 148–151 (1941). [MF 16354]

For a given space curve an affine parameter can be defined so that $x'=t$, $t'=p$, $p'=\kappa_1 t+w$, $w'=\kappa_2 t+3\kappa_1 p$, where x, t, p, w are, respectively, the position vector, tangent vector, and affine principal normal and binormal of Winternitz, and κ_1, κ_2 are the first and second affine curvatures. The developable surface generated by the above binormals (affine binormal surface of Winternitz) is studied, and it is proved that the space curve is a flecnode curve on this surface if κ_1 is constant and nonzero, this condition being also necessary. The equation of the osculating quadric of the above developable surface is obtained.

A. G. Walker (Liverpool).

Prostschin, M. Ueber einige Regelflächen. Nauk.-Doslid. Inst. Mat. Meh. Harkiv. Univ. Geometričniĭ Zbirnik **2**, 29–32 (1940). (Russian. German summary) [MF 16943]

Schraubenregelfläche heisst eine Fläche, beschrieben von der Geraden BP, welche den Kreiszylinder in den Punkten einer Schraubenlinie berührt und mit der zur Zylinderaxe normalen Ebene konstanten Winkel bildet. Die Fläche hat die Eigenschaften: (1) die Strictionslinie der Fläche fällt mit der genannten Schraubenlinie zusammen, (2) die Zentralebene ist die Tangentialebene des Zylinders längs der betreffenden Erzeugenden. Diese Eigenschaften bestehen, wenn man einen beliebigen Zylinder und eine beliebige darauf liegende Kurve nimmt. *Author's summary.*

Erim, Kerim. Die höheren Differentialelemente einer Regelflaeche und einer Raumkurve. Rev. Fac. Sci. Univ. Istanbul (A) **10**, 1–24 (1945). (German. Turkish summary) [MF 15966]

The author gives a geometrical representation for the "element of contact of order p" of a ruled surface. To this end he considers the moving trihedron of the surface, consisting of the generator, the normal to the surface at the central point, and the normal to these two. The instantaneous axis of rotation of this trihedron, called "axis of curvature," generates another ruled surface. The process can be iterated, and the author shows that the generator and the first $p-1$ successive axes of curvature determine the element of contact of order p. The analytical tool is E. Study's representation of ruled surfaces as curves on the unit sphere in the space of dual vectors. The second part contains an analogous development for real spherical curves;

it introduces the concept of spherical center of curvature, defined as the center (on the sphere) of a circle on the sphere, having 3-point contact with the given curve.
H. Samelson (Ann Arbor, Mich.).

Urisman, S. L'involution et la théorie des surfaces. Nauk.-Doslid. Inst. Mat. Meh. Harkiv. Univ. Geometričnii Zbirnik 2, 85–92 (1940). (Russian. French summary) [MF 16946]

L'auteur montre que plusieurs questions au premier aspect différentes de la théorie des surfaces se ramènent de fait à l'application de la théorie de l'involution des formes du premier degré, si l'on considère les deux formes quadratiques fondamentales comme équations des éléments doubles des involutions correspondantes. *Author's summary.*

Vincensini, Paul. Sur la déformation des surfaces. C. R. Acad. Sci. Paris 222, 630–632 (1946). [MF 16042]

Congruences C of lines l lying in the tangent planes of a surface S at its points M but not passing through M are sought with the following property: the lines joining M to the focal points of l separate harmonically the tangents to the curves of a net R invariant under arbitrary deformations of S. Two types of such congruences are found, one being the congruences of Ribaucour (the focal points lying on conjugate tangents to S); the second type of congruence satisfying the conditions has for its net R a degenerate net composed of a one parameter family of curves, and one of the focal points of l lies on the tangent to that curve. That focal point persists under arbitrary deformations of S. This is a characteristic property of such congruences.
V. G. Grove (East Lansing, Mich.).

Vincensini, Paul. Congruences arbitrairement déformables avec fixité des points centraux sur les différents rayons. C. R. Acad. Sci. Paris 222, 1326–1328 (1946). [MF 16737]

The problem proposed is the determination of a surface Σ generated by a point M and a congruence C of lines d lying in the tangent plane to Σ at M such that, under an arbitrary deformation of Σ, the midpoints of the focal points of C remain fixed on their respective lines d. It is found that Σ may be chosen arbitrarily and that C is a congruence of Ribaucour, the focal points of whose lines are situated on conjugate tangents of Σ. *V. G. Grove.*

Gauthier, Luc. Les congruences linéaires de l'espace projectif à R dimensions. Revue Sci. (Rev. Rose Illus.) 82, 347–358 (1944). [MF 16475]

Expository article.

Darmostuck, P. Über Geradenkomplexe deren Komplexkugel eine Fläche nach einer von einem oder zwei Parametern abhängenden Kurvenfamilie schneiden. Nauk.-Doslid. Inst. Mat. Meh. Harkiv. Univ. Geometričnii Zbirnik 2, 11–28 (1940). (Russian. German summary) [MF 16942]

Nimmt man auf der Fläche (1) $x=f(u,v)$, $y=g(u,v)$, $z=h(u,v)$ eine Kurvenfamilie (2) $u=u(v,c,c_1)$, welche von zwei Parametern c, c_1 abhängt, und im Raume eine andere Kurvenfamilie (3), welche von denselben Parametern c, c_1 abhängt, und verbindet man jeden Punkt von (3) mit allen Punkten der denselben Werten der Parameter c, c_1 ensprechenden Kurve von (2), so bildet das so entstandene System der Geraden einen Komplex dann und nur dann, wenn (3) eine isotrope Kongruenz der Normalen ist. Die Gleichung des betreffenden Komplexes wird abgeleitet.
From the author's summary.

Renaud, Paul. Représentation de la convergence de trajectoires quelconques sous l'influence d'une déformation. J. Phys. Radium (8) 6, 265–271 (1945). [MF 16327]

The mapping of a surface S_1 on a surface S_2 by a congruence of rays is in general locally linear; in degenerate cases we get a line image (when the Jacobian vanishes) or a point image (when all the linear coefficients vanish). These ideas are applied to the problem of the correction of defective vision, in particular, the case of a cornea of complex curvature. The deformation mentioned in the title is the deformation of the congruence due to the insertion of a correcting lens. *J. L. Synge* (Pittsburgh, Pa.).

Bouligand, Georges. Sur la théorie des surfaces applicables. C. R. Acad. Sci. Paris 222, 263–265 (1946). [MF 15996]

The following is an extract from the author's introduction. D'après Voss, deux surfaces S_0, S_1 sont dites applicables s'il existe une famille continue à un paramètre t de surfaces S_t isométriques entre elles quel que soit t, et telles que S_t coïncide avec S_0 pour $t=0$, avec S_1 pour $t=1$. Notre objet principal est le théorème suivant, greffé sur les examples classiques d'obtention de surfaces ayant un ds^2 donné au moyen de quadratures.

Soit E un ensemble de surfaces $M=M(u,v)$ à courbures continues sur chacune desquelles l'arc image d'un arc rectifiable du plan (u,v) est exprimé par l'intégrale d'un élément $\sqrt{(edu^2+2fdudv+gdv^2)}$, telles en outre que les quadratures donnant $M(u,v)$ portent sur des fonctions (déduites de e,f,g) où interviennent des signes $\pm$ en nombre fini. En remplaçant chaque $\pm$ par une fonction mesurable de la variable d'intégration, on passe de E à un nouvel ensemble E^* résolvant un problème restreint d'isométrie dans le champ des surfaces rectifiables. Entre une paire de surfaces isométriques, s'offrant comme couple isolé dans E, il existe une applicabilité par l'entremise des surfaces de E^*.
P. Scherk (Saskatoon, Sask.).

Llensa, Georges. Attributs de dérivabilité dans la génération de systèmes triples orthogonaux. C. R. Acad. Sci. Paris 222, 845–847 (1946). [MF 16279]

The author continues his study of triply orthogonal systems based on the differential equation (*) $\text{grad}^2\, u\cdot S^2(u)=1$, where $S(u)$ is the left side of the equation of a sphere with radius and center depending on u. He considers a contact transformation connected with a solution of (*) with spherical level surfaces and constructs triply orthogonal systems for which the second derivatives do not exist.
H. Samelson (Ann Arbor, Mich.).

Mihăilescu, Tiberiu. Sur les réseaux conjugués à surfaces associées coïncidantes. Bull. Math. Soc. Roumaine Sci. 46, 43–75 (1944). [MF 16509]

Let A_0 describe a conjugate net (A_0) on a surface in a projective space of three dimensions, and let A_1, A_2 be the Laplace transforms of A_0. Let A_3 be the harmonic conjugate of A_0 with respect to the two focal points on the axis of (A_0). The locus of A_3 and the envelope of the plane (A_1, A_2, A_3) are then called associate surfaces. This paper studies nets (A_0) whose associate surfaces coincide. There are two types of such nets; in one of these (A_0) is harmonic. If (A_0) is not harmonic it depends on five arbitrary functions of one variable; for such nets there exists an infinity of points on the ray of the net generating surfaces whose tangent planes pass through the axis of (A_0). If (A_0) is harmonic, it depends

on six arbitrary functions of one variable; the point A_3 generates a conjugate net (A_3); the line joining A_3 to A_1 (or A_2) generates a congruence whose focal points are A_1 (or A_2), A_3 and the developables of the two congruences so generated correspond; the net (A_0) then admits a periodic (period four) sequence of Laplace. Conversely, the only conjugate nets admitting a periodic (period four) sequence of Laplace are harmonic nets whose associate surfaces coincide. For such nets the ray and axis congruences are W. Conversely if the ray and axis congruences of a harmonic net are W, the associate surfaces coincide. The paper concludes by the determination of conjugate nets whose ray and axis are reciprocal polars with respect to a fixed quadric. Except for conjugate nets on a quadric surface, such nets are nets of Darboux-Tzitzeica, and admit a periodic (period four) sequence of Laplace. The method used is that of É. Cartan. *V. G. Grove* (East Lansing, Mich.).

Ling, Donald P. Geodesics on surfaces of revolution. Trans. Amer. Math. Soc. **59**, 415–429 (1946). [MF 16464]

Let a surface S be generated by revolving about Oy a curve C of class C' which rises monotonically from the origin to infinity. On every geodesic g of S, other than a meridian, there lies a point P such that the plane determined by P and the axis of S is a plane of symmetry of g. All intersections of g with itself lie in this plane and may be numbered from P as the first, second, etc., double points of g. It is shown that there exists a (finite or infinite) sequence of parallels dividing S into zones such that to every point Q of the nth zone there correspond geodesics having Q as the kth double point for $k=1, 2, \cdots, N+n-2$, but no such geodesic for $k>N+n-2$, where N is an integer depending on the slope of C at O. With a geodesic on S may be associated the number of double points or, equivalently, the number of revolutions it makes about the axis of S. An admissible surface S is called "cylindrical," "conical," or "semicylindrical" according as (a) the number of revolutions for each geodesic g of S is infinite, (b) the number of revolutions is bounded for all geodesics of S, or (c) the number of revolutions is finite for each g but not bounded for all g on S. All admissible surfaces belong to one of these types. Examples are given of each type, and criteria are established for determining the type from the slope function of C. Finally, it is shown that the indicated results are valid for certain surfaces S where the curve C is not of class C'. *S. B. Jackson* (College Park, Md.).

Kasner, Edward, and De Cicco, John. Multi-isothermal systems. Revista Unión Mat. Argentina **11**, 117–125 (1946). (Spanish) [MF 16368]

Ein System von ∞^{2n-1} Kurven des Raumes R_{2n} von n komplexen Veränderlichen heisse "multi-isotherm," falls es sich durch eine pseudokonforme (d.h. analytische) Abbildung in ein Bündel von ∞^{2n-1} parallelen Geraden überführen lässt. In ähnlicher Weise wird ein "multi-isothermes" System von ∞^1 Hyperflächen des R_{2n} als pseudokonformes Bild von ∞^1 parallelen Hyperebenen definiert. Es wird zunächst bewiesen, dass der von irgendeinem multi-isothermen System von Hyperflächen und einem beliebigen multi-isothermen Kurvensystem gebildete "Pseudowinkel" eine multi-harmonische Funktion ist (der Pseudowinkel ist invariant gegenüber pseudokonformen Abbildungen). Es besteht ferner der Satz, dass eine konforme (analytische) Fläche jedes multi-isothermes Hyperflächensystem in einem isothermen System von Kurven schneidet. Es gelten auch die beiden Umkehrungen dieses Satzes; d.h. einerseits ist eine Fläche notwendig konform, falls sie durch jedes multi-isotherme Hyperflächensystem in einem isothermen System von Kurven geschnitten wird. Andererseits ist auch ein System von ∞^1 Hyperflächen stets dann multi-isotherm, falls es mit jeder konformen Fläche je ein isothermes Kurvensystem gemeinsam hat. Die vorstehenden Ergebnisse wurden von den beiden Verfassern in einer früheren Arbeit für den R_4 bewiesen [Bull. Amer. Math. Soc. **51**, 169–174 (1945); diese Rev. **6**, 186]. *P. Thullen.*

Su, B. On certain tac-invariants of two curves in a projective space. Quart. J. Math., Oxford Ser. **17**, 116–118 (1946). [MF 16778]

The author gives another proof of his theorem that two curves in a projective n-dimensional space which have a point and the first r osculating spaces defined at this point in common have $r-1$ projective invariants [cf. Acad. Sinica Science Record **1**, 16–19 (1942); these Rev. **5**, 14]. This result has recently been extended by Segre [Quart. J. Math., Oxford Ser. **17**, 35–38 (1946); these Rev. **7**, 393]. *J. E. Wilkins, Jr.* (Buffalo, N. Y.).

Satô, Saburô. Die projektive Differentialgeometrie als eine Verallgemeinerung der N. E. Differentialgeometrie. III. Flächentheorie. Tôhoku Math. J. **48**, 89–147 (1941).

[For part II see the same J. **46**, 181–233 (1940); these Rev. **2**, 21.] The author considers a surface S in the three-dimensional projective space with the property that with each point P is connected a quadric $Q(P)$, which does not pass through P and does not touch the tangent plane at P. By means of these quadrics it is possible to define (1) a metric on S; (2) a normal at P (the line connecting the point P with the pole of the tangent plane at P with respect to Q); (3) at each point P a sphere tangent to S at P; this quadric is a sphere in the sense of the non-Euclidean geometry based on $Q(P)$; (4) a Clifford surface at P which is tangent to S; (5) lines of curvature; (6) the quadrics of Darboux, Lie and Wilczynski. If the quadric Q is chosen independent of P, one obtains the differential geometry of non-Euclidean spaces. The theory leads, however, to the projective differential geometry if the quadric Q is a projective covariant of the surface. Such a quadric is obtained by some requirements, for example, that the Lie quadric shall be identical with the tangent sphere mentioned above and the normal at P identical with Wilczynski's projective normal. Geometrical interpretations are given for the invariants obtained in the theory. The quadric Q can also be chosen in such a way that the affine theory is obtained. *J. Haantjes* (Amsterdam).

Wong, Yung-Chow. Scale hypersurfaces for conformal-Euclidean space. Amer. J. Math. **68**, 263–272 (1946). [MF 16423]

The author applies to space of n dimensions some of the definitions and theorems obtained by Kasner and De Cicco [Amer. J. Math. **67**, 157–166 (1945); these Rev. **6**, 186]. Let C_n be an n-space which is represented conformally on a Euclidean n-space R_n. A scale hypersurface is the locus of a point along which the scale function $\sigma=dS/ds$ of the corresponding differentials of arc lengths of C_n and R_n does not vary. As defined in the paper quoted above, a system of ∞^1 hypersurfaces is quasi-isothermal if they represent the scales of a conformal map of a C_n on an R_n such that the curvature of C_n is constant along any one of these scale hypersurfaces. The following extensions to n-space of cer-

tain theorems of Kasner and De Cicco are proved. (1) All real quasi-isothermal families of hypersurfaces of the parallel type are (a) parallel hyperplanes, (b) concentric hyperspheres and (c) generalized coaxial cylinders of rotation. (2) The only quasi-isothermal families of hyperplanes (hyperspheres for $n>2$) are the pencils (the concentric sets). (3) A family of generalized cylinders of rotation which are generated from a family of ∞^1 hyperspheres in an r-plane is quasi-isothermal if and only if it is a pencil of cylinders for $n>2$ and a pencil of circles for $n=2$. The form of the linear-element of C_n is obtained for each of the cases outlined in all these theorems. *J. De Cicco* (New York, N. Y.).

Pimiä, Lauri. Abbildung der Lie'schen Kugelgeometrie auf eine höhere komplexe Gerade. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. 4, 50 pp. (1941). [MF 16498]

Let $\alpha=a_0+a_1e_1+a_2e_2+a_3e_3$ be a quaternion with complex coefficients (the units e_1, e_2, e_3 satisfying $e_1^2=e_2^2=e_3^2=e_1e_2e_3=-1$). Stephanos [Math. Ann. 22, 589–592 (1885)] represented α by the sphere with center (a_1, a_2, a_3) and radius ia_0, so that the condition for spheres α and β to touch is the vanishing of the norm $N(\alpha-\beta)$. Johansson [Neuvième Congrès Math. Scand., Helsingfors, 1938, pp. 171–178] found that there is some advantage in replacing α by $\xi_1\bar{\xi}_2$ (that is, the product of one quaternion and the conjugate of another), so that the sphere

$$N(\xi_2)(x^2+y^2+z^2)-2a_1x-2a_2y-2a_3z+N(\xi_1)=0$$

represents a point (ξ_1, ξ_2) of the hypercomplex projective line [Cartan, Leçons sur la Géométrie Projective Complexe, Paris, 1931.] When $N(\xi_1)=0$ we have a sphere through the origin; when $N(\xi_2)=0$ we have a plane. The author develops the theory of contact transformations of the spheres and the corresponding collineations and correlations of the hypercomplex line. Spheres having two common points form a "bundle" (lineare Kugelschar) and represent a "simple chain" of points on the hypercomplex line; such spheres or points (ξ_1, ξ_2) satisfy an equation of the form

$$\xi_1\lambda\bar{\xi}_2+\xi_2\mu\bar{\xi}_2-\xi_1\nu\bar{\xi}_1-\xi_2\bar{\lambda}\bar{\xi}_1=0, \quad \mu+\bar{\mu}=\nu+\bar{\nu}=0.$$

On the other hand, spheres which satisfy the equation

$$\lambda\xi_2\bar{\xi}_1+\xi_1\bar{\xi}_2\bar{\lambda}-n\cdot N(\xi_1)+m\cdot N(\xi_2)=0$$

form a "linear complex" of spheres and represent a "three-dimensional chain" of points. *H. S. M. Coxeter.*

Pimiä, Lauri. Über die linearen Kugelkomplexe bei der involutorischen Berührungstransformation $\mathfrak{T}$. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. 16, 11 pp. (1943). [MF 16502]

The transformation $(\xi_1, \xi_2)\to(\eta_1, \eta_2)$, where $\alpha\xi_1+\beta\xi_2=\eta_1\sigma$, $\gamma\xi_1+\delta\xi_2=\eta_2\sigma$, $N(\sigma)\neq0$ [see the preceding review] is involutory if $\alpha^2+\beta\gamma=\gamma\beta+\delta^2=1$ and $\alpha\beta+\beta\delta=\delta\gamma+\gamma\alpha=0$. The author determines the "linear complexes" of spheres which are invariant under this transformation.

H. S. M. Coxeter (Toronto, Ont.).

Pimiä, Lauri. Die bei der involutorischen Berührungstransformation $\mathfrak{T}$ invarianten Kugelscharen. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. 21, 8 pp. (1943). [MF 16503]

Continuing his study of the involutory transformation defined in the preceding review, the author determines the invariant "bundles" of spheres. *H. S. M. Coxeter.*

Hlavatý, Václav. Zur Lie'schen Kugelgeometrie. I. Kanalflächen. Věstník Královské České Společnosti Nauk. Třída Matemat.-Přírodověd. 1941, 30 pp. (1941). [MF 16121]

In der Lieschen Kugelgeometrie handelt es sich um die projektiven Eigenschaften eines 5-dimensionalen Raumes, in welchem eine 4-dimensionale Quadrik Q, die die Menge der Kugeln darstellt, gegeben ist. In dieser Arbeit wird die Kanalfläche, d.h. die Kurven auf Q, eingehend untersucht. Die Liesche Oskulationszyklide wird definiert. Auch werden Frenet'sche Formeln aufgestellt. In dieser Formeln treten drei Krümmungen auf, die als Funktionen des Projektivbogens die Kanalfläche charakterisieren.

J. Haantjes (Amsterdam).

Tonolo, A. Sulle ipersuperficie V_n le cui due forme fondamentali ammettono una stessa trasformazione infinitesima $X(f)$. Boll. Un. Mat. Ital. (2) 5, 137–140 (1943). [MF 16098]

The author proves the following theorem. If the two fundamental forms of a V_n in R_n admit the same infinitesimal transformation, there exists at least one relation between the principal curvatures. A V_2 in R_3 with this property is a W-surface. *J. A. Schouten* (Epe).

Droste, J. The concept "reduced length" in a space of N dimensions. Nederl. Akad. Wetensch. Verslagen, Afd. Natuurkunde 53, 269–273 (1944). (Dutch. German, English and French summaries) [MF 15788]

If (θ, u) is a geodesic polar coordinate system on a surface with pole at A the line element has the form $ds^2=du^2+Gd\theta^2$. The value of $\sqrt{G}$ at P is called the reduced length of the geodesic AP. It satisfies the differential equation

$$(1) \qquad d^2y/dx^2+ky=0,$$

where k is the total curvature of the surface. The author defines the notion of reduced length in Riemannian spaces of n dimensions as follows. If $\theta^1, \cdots, \theta^{n-1}$ is a system of geodesic polar coordinates with pole at A we have

$$ds^2=du^2+a_{pq}d\theta^p d\theta^q.$$

The reduced length of AP is defined as

$$(AP)=(\sqrt{a})_P/\{\lim_{u\to0}\sqrt{a}/u^{n-1}\}, \quad a=\operatorname{Det} a_{pq}.$$

The equations which are analogous to (1) turn out to be

$$\frac{\delta X^h}{du}=Y^h; \quad \frac{\delta Y^h}{du}=R_{kji}{}^hX^k\frac{dx^j}{du}\frac{dx^i}{du},$$

where $R_{kji}{}^h$ are the components of the curvature affinor with respect to the coordinate system x^h. The author gives n solutions of these equations by means of which he proves the theory of symmetry, namely, $(AB)=(BA)$.

J. Haantjes (Amsterdam).

Shapiro, J. On arbitrary components of a tensor of rank 2. Rec. Math. [Mat. Sbornik] N.S. 17(59), 65–84 (1945). (Russian. English summary) [MF 14594]

The metric tensor g_{ij} of an n-dimensional Riemannian space has $n(n+1)/2$ components which are arbitrary functions, but the arbitrariness is due in part to the freedom of choice of the coordinate system. The first part of this paper contains a rigorous proof of the proposition, first stated by Riemann, that it takes $n(n-1)/2$ arbitrary functions to determine a space; this is proved by showing that there exists a coordinate transformation which takes a given metric tensor into one in which the diagonal elements g_{ii} have preassigned values; this, in turn, is based on a lemma

dealing with existence and uniqueness of solutions of a system of partial differential equations, a lemma generalizing known propositions. The central concept in the rest of the paper is that of a net, that is, a set of n families of curves in a Riemannian n-space. In addition to nets which correspond to coordinate systems the author considers two types of nets generalizing properties previously introduced for nets on a surface: a Čebyšev net is a holonomic net (that is, one produced by n families of hypersurfaces) which has the further property that two hypersurfaces of one family cut out arcs of equal lengths on all curves of the family they intersect; a "net of equal paths" is an oriented net (that is, one in which a positive direction has been chosen on each family) which has the further property that there exists a function ϕ such that $ds=d\phi$ as we move in the positive direction on each curve of the net. In connection with a net the author considers a family of n vector fields consisting of unit vectors tangent to the curves of the net; two of these fields determine at each point by means of a Poisson bracket expression a "Poisson vector"; another vector used is the "axial vector" of n vectors at a point; it forms the same angle with all these vectors and its length is the reciprocal of the cosine of this angle. In terms of these vectors different properties of nets are established; for example: a condition for a net to be a Čebyšev net is that it can serve as a coordinate net with all g_{ii}'s having the value one, or that all its Poisson vectors vanish; a condition for a net to be of equal paths is that the field of its axial vectors is a gradient field; a Čebyšev net is a net of equal paths for any choice of orientation. *G. Y. Rainich.*

Bompiani, E. Intorno alle varietà isotrope. Ann. Mat. Pura Appl. (4) 20, 21–58 (1941). [MF 16596]

The main point at issue is the relationship between the theory of isotropic varieties as developed by Lense and Pinl [for example, J. Lense, Math. Ann. 116, 297–309 (1939)] and that of Riemannian geometry of higher order as developed by the author [for example, R. Accad. Italia. Mem. Cl. Sci. Fis. Mat. Nat. (8) 6, 269–520 (1935)]. Essentials of the latter discipline are given in the early pages of the present paper. The author's principal interest is in showing the desirability of adopting a projective viewpoint in exploiting the above relationship. He illustrates this thesis by a detailed study of specific problems. They are: (1) the classification of isotropic ruled surfaces, (2) the further investigation of some isotropic surfaces and varieties studied by Lense and Pinl and (3) the derivation of theorems regarding the projection of an isotropic variety on certain associated linear spaces. *J. L. Vanderslice.*

Botella Raduan, F. On the fundamentals of the introduction to the spaces with affine and projective connection. Revista Mat. Hisp.-Amer. (4) 6, 17–24 (1946). (Spanish) [MF 16706]

This is an exposition of the theories mentioned in the title without reference to the original papers of Cartan, Veblen and Whitehead, who developed them. The topics treated include coordinate manifolds, associated geometric objects and groups of transformations, and applications to affine, metric, and projective spaces. *C. B. Allendoerfer.*

Maxia, A. Varietà anolonome immerse in una varietà a connessione affine. (X_n^{n-1} in E_n affine.) Mem. Soc. Roy. Sci. Lett. Bohême. Cl. Sci. 1939, 18 pp. (1939). [MF 16127]

The first part of this note was published in Časopis Pěst. Mat. Fys. 68, 33–49 (1939). In this second part the special case of an X_n^{n-1} in E_n is discussed. In order to get an induced connexion the X_n^{n-1} has to be rigged (eingespannt). In the holonomic case there is only one pseudonormal direction but in the case of an X_n^{n-1} two different invariant methods of rigging exist. The first leads to an induced connexion already discussed by V. Hlavatý [Math. Z. 38, 283–300 (1934)]. The second leads to another induced connexion. The author gives the relations between the two pseudonormal vectors and a geometric interpretation of the second induced connexion for $n=3$. *J. A. Schouten* (Epe).

Su, Buchin. A note on the projective differential geometry of a non-holonome surface. Ann. Mat. Pura Appl. (4) 20, 213–220 (1941). [MF 16603]

The author establishes several theorems concerning the curves in which a nonholonomic surface is intersected by planes through a nonasymptotic tangent. A typical result is that there are in general three planes through such a tangent which produce sections which are hyperosculated by their osculating conics. This is in contrast to the holonomic case in which there are only two such planes.
J. E. Wilkins, Jr. (Buffalo, N. Y.).

Freeman, J. G. Theory of a ruled two-space in a generalized metric space. Quart. J. Math., Oxford Ser. 17, 119–128 (1946). [MF 16779]

The generalized metric space S_n in question is an n-dimensional metric space to each point of which is associated a contravariant vector density called the element of support. A surface S_2 containing a one parameter family of generalized extremals of S_n (that is, curves with first variation of length integral vanishing for arbitrary displacement of both curve and element of support) is called a ruled surface with the extremals as generators. The author investigates the case in which the element of support is tangential to the generators, utilizing some results from a previous paper [same J. 15, 70–83 (1944); these Rev. 6, 188]. The Gauss equation for S_2 leads to curvature properties generalizing corresponding properties of Riemannian and Euclidean space. The notion of central point and point of striction exists here, as does a direct generalization of Bonnet's theorem regarding the line of centers. *J. L. Vanderslice.*

Galvani, Octave. Sur la réalisation des espaces de Finsler. C. R. Acad. Sci. Paris 222, 1067–1069 (1946). [MF 16522]

The following theorem is proved. The local realization of an n-dimensional Finsler space can be made by means of an n-dimensional point variety σ at each point M of which is attached a linear element L centered at M and an n-plane passing through L. This configuration lies in a Euclidean space of $2n^2-n$ dimensions. If L is always tangent at M to σ, the space is a Riemann space. This result is a generalization of a theorem of É. Cartan which described the realization of certain Finsler spaces by means of linear elements lying in a Euclidean space of $2n-1$ dimensions.
C. B. Allendoerfer (Haverford, Pa.).

Galvani, Octave. Les connexions finslériennes de congruences de droites. C. R. Acad. Sci. Paris 222, 1200–1202 (1946). [MF 16728]

The author continues his treatment of Finsler spaces [see the preceding review] with a discussion of Finsler planes (two dimensional) which can be realized in Euclidean three-space E_3. A realization in E_3 is possible for a Finsler plane of zero torsion when and only when it admits an absolute

parallelism of linear elements. The realization is in terms of a set $S=(M, \Delta, P)$, where M is a point, Δ is a line through M, and P is a plane passing through Δ. An arbitrary set S is a realization of a Finsler plane only if the congruence Δ satisfies certain conditions. Various other properties of Δ are developed. *C. B. Allendoerfer* (Haverford, Pa.).

Mikami, Misao. Geometry of the integral $\int(A_i x^{(m)i}+B)^{1/p}dt$. Jap. J. Math. 18, 663–673 (1943). [MF 14978]

Si l'on définit, sur une variété à n dimensions, une métrique par une intégrale

$$(\alpha) \qquad s=\int F(x^i, dx^i/dt, \cdots, d^m x^i/dt^m)dt,$$

invariante par les transformations des coordonnées (x^i) et par les changements du paramètre t, on dit que cette variété constitue un espace de Kawaguchi d'ordre m. Ce papier est consacré à l'étude de l'espace de Kawaguchi particulier associé à

$$(\beta) \qquad s=\int (A_i d^m x^i/dt^m+B)dt,$$

où les A_i et B sont des fonctions de l'élément de courbe d'ordre $m-1$. Dans le cas général (α) Hombu [Mem. Fac. Sci. Kyûsyû Imp. Univ. A. 1, 29–110 (1940); ces Rev. 2, 23] a montré qu'il existait une classe projective privilégiée de "paths" d'ordre $m+1$ et qu'à partir de ce système de paths on pouvait définir, sur la variété des éléments de courbe d'ordre m, une connection linéaire. En modifiant le $(m-1)^*$ vecteur de Synge [Amer. J. Math. 57, 679–691 (1935)], l'auteur construit dans le cas (β) un système privilégié de paths d'ordre m, invariants à la fois par les transformations de coordonnées et les changements de paramètre, puis à partir de ce système une connexion linéaire. Les paths considerés coincident avec les courbes minimales de la métrique (β). Cette étude diffère assez sensiblement de celle donnée par Hokari [J. Fac. Sci. Hokkaido Imp. Univ. Ser. I. 8, 63–78 (1940); ces Rev. 2, 22] pour le même problème et se rapproche davantage de celle de Kawaguchi [Trans. Amer. Math. Soc. 44, 153–167 (1938)] pour le cas particulier de l'intégrale (β) avec $m=2$.

A. Lichnerowicz (Strasbourg).

NUMERICAL AND GRAPHICAL METHODS

***Jahnke, Eugene, and Emde, Fritz. Tables of Functions with Formulae and Curves.** 4th ed. Dover Publications, New York, N. Y., 1945. xv+306+76 pp. $3.75.

This edition differs from that of 1943 [same publisher; cf. these Rev. 4, 281, 340] principally by the correction of nearly 400 errors and the inclusion of three pages of supplementary bibliography by R. C. Archibald.

***Turrell, Franklin Marion. Tables of Surfaces and Volumes of Spheres and of Prolate and Oblate Spheroids, and Spheroidal Coefficients.** University of California Press, Berkeley and Los Angeles, 1946. xxiii+153 pp. $2.00.

The spheroids mentioned in the title are the surfaces obtained by revolving an ellipse about its major or minor axis, respectively. The tables were constructed to facilitate research connected with citrus fruit. The main table gives the surfaces and volumes to three significant figures of which the last is not quite safe; the tables are arranged according to the difference $\delta=2a-2b$ of the axes which varies from 0 to 3 in steps of .1. For each value of δ the tables are arranged with $2a$ as argument, which varies in steps of .1 up to 15, starting from a value for which the ratio b/a is nearly .5. The "spheroidal coefficients" are the ratios of the volumes or surfaces of spheroids to those of spheres of the same diameter. These tables are arranged with b/a as argument, running in steps of .01 from 0 to 1. *W. Feller.*

Strömgren, Bengt. Optical sine-tables giving seven-figure values of $x-\sin x$ with arguments x and $\sin x$. Mém. Inst. Géodésique Danemark [Geodætisk Instituts Skr.] (3) 5, 63 pp. (1945). [MF 16271]

The tabulation by Chrétien [Nouvelles Tables des Sinus . . . , Revue d'Optique, Paris, 1932] of $x-\sin x$ (in radians) to five decimals up to about 30° and to six decimals to 10° is here extended to seven decimals, for use in high-accuracy trigonometrical ray-tracing. The tables are not in critical form (as in Chrétien) but have the very fine argument of 0.0001, up to 0.5000 in each table. Interlinear differences (the maximum being 155) and proportional parts are given throughout. A supplementary table gives 7-figure values of $\tan x$ for $x=0(0.001)0.200$, with interlinear differences; this table is linear.

The tables have been prepared from those in the British Association Mathematical Tables, vol. 1 [Cambridge University Press, 1931; cf. these Rev. 7, 337], extreme care being taken to ensure the accuracy of the last decimal. Examples of the application of the tables to the object glass of a telescope are given. *L. J. Comrie* (London).

Kaplan, E. L. Auxiliary table of complete elliptic integrals. J. Math. Phys. Mass. Inst. Tech. 25, 26–36 (1946). [MF 16148]

The difficulty of tabulating the complete elliptic integrals as k^2 approaches 1 is well known. As K and E approach their limits logarithmically, the author tabulates them with argument $\log(1-k^2)$ and thus produces tables that are easily interpolated with Bessel's formula, for which mean second differences are provided. Ten decimals are given. The advantage of this form of tabulation becomes apparent as k^2 approaches 0.999 999, the limit of the table, where the second difference almost vanishes.

The author has checked Hayashi's tables, which, he rightly says, "are notorious for their inaccuracy." He discovered four errors in K, not knowing that Hayashi himself had published them in 1932, and a fifth already given in Scripta Math. 3, 365 (1935). He also found six in E, all of which (and no more) were found by the reviewer in 1933, by differencing. Of Milne-Thomson's 9-figure table he says that the last figure is unreliable; this statement is somewhat misleading. Actually, the reviewer's records show about 70 errors (in 200 values) of a unit each, one of two units, and none greater. Two photographic reprints of Legendre's tables are mentioned, but not that by Potin [Paris, 1925].

L. J. Comrie (London).

Kerridge, Siegfried. Anwendung der Nationalbuchungsmaschine für wissenschaftliche Rechnungen. Z. Angew. Math. Mech. 21, 242–249 (1941). [MF 15856]

This paper consists of literal translations from an article by L. J. Comrie [J. Roy. Statist. Soc., Supplement 3, no. 2, 87–113 (1936)]. It describes the application of the machine

to three of the main problems of finite differences, namely, summation from finite differences (up to the sixth), differencing (to the fifth difference) and subtabulation, or systematic interpolation to smaller intervals such as fifths or tenths, with printing of all results.

It is of interest to note that this machine, which has led to a revolution in table-making methods, has been much more appreciated in England and in Germany than in the U. S. A., where it is made. *L. J. Comrie* (London).

Pedersen, Peder. Über die numerische Berechnung der Kettenbrüche nebst einer Berechnung der Grundzahl der natürlichen Logarithmen. Geodætisk Institut, København, Meddelelse no. 14, 36 pp. (1940). [MF 16184]

Pedersen, Peder. Berechnung der Grundzahl e der natürlichen Logarithmen mit 606 Dezimalen. Geodætisk Institut, København, Meddelelse no. 16, 17 pp. (1942). [MF 16183]

Pedersen, Peder. Fortsetzung der Berechnung der Grundzahl e der natürlichen Logarithmen bis zur 808. Dezimalstelle. Geodætisk Institut, København, Meddelelse no. 17, 21 pp. (1944). [MF 16182]

These three papers give the results of calculating the number e to 404, 606, and 808 decimal places, respectively. All calculations are based on Euler's continued fraction

$$\text{(1)}\qquad \tfrac{1}{2}(e-1)=\frac{1}{1+}\,\frac{1}{6+}\,\frac{1}{10+}\cdots\frac{1}{+4n+2+}\cdots.$$

The digits of e are determined one at a time by multiplying a previously determined value of e by the denominator B_n of the convergent A_n/B_n of (1), annexing those digits which will make the product nearly an integer. The first paper gives a theoretical discussion of this general method. The papers also contain the values of B_n for $n\leqq 170$. These are given by

$$B_1=1,\quad B_2=7,\quad B_3=71,\ \cdots,\ B_{n+1}=(4n+2)B_n+B_{n-1};$$

B_{170} is a 408 digit number. *D. H. Lehmer.*

Ferguson, D. F. Value of π. Nature 157, 342 (1946). [MF 15812]

This note reports that a new calculation of π seems to indicate an error in Shanks's [Proc. Roy. Soc. London 22, 45–46 (1873)] long accepted 707 decimal place value beyond the 527th decimal. The author used the formula

$$\pi/4=\operatorname{arccot} 1=3\operatorname{arccot} 4+\operatorname{arccot} 20+\operatorname{arccot} 1985,$$

which he attributes to R. W. Morris. Shanks used the more efficient formula of Machin,

$$\operatorname{arccot} 1=4\operatorname{arccot} 5-\operatorname{arccot} 239.$$

[The author's calculated value through the 620th decimal is reproduced in Math. Tables and Other Aids to Computation 2, 144–145 (1946).] *D. H. Lehmer.*

Tricomi, Francesco. Generalizzazione di una formula asintotica sui polinomi di Laguerre e sue applicazioni. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. 76, 288–316 (1941). [MF 16268]

An approximation to

$$L_n^{(\alpha)}(t)=\frac{1}{n!}e^t t^{-\alpha}\frac{d^n}{dt^n}(e^{-t}t^{n+\alpha})$$

for $0<t<4n$ is derived by means of the saddle-point method. The result is

$$\text{(1)}\quad (\pi n\sin 2\theta)^{-\frac{1}{2}}(2\sin\theta)^{-\alpha}e^{\frac{1}{2}t}\times\cos\{n(2\theta+\sin 2\theta)+(1+\alpha)\theta-(1+2\alpha)\pi/4\},$$

where $\sin\theta=(t/4n)^{\frac{1}{2}}$. From this result Tricomi deduces a corresponding one for Hermite polynomials. A numerical comparison shows that in the case of H_7 Tricomi's approximation is better than that of Plancherel and Rotach [Comment. Math. Helv. 1, 227–254 (1929)]. The approximation (1) is a good one even for small values of n and is used by the author for a study of the zeros of Laguerre polynomials.

Numerical material in the paper includes a short table, to 4 decimals, of the roots of the transcendental equation $\xi+\sin\xi=a$ for $a=0.0(0.1)3.0(0.02)3.18$, a graph of the zeros of $L_n(t)$ for $n=1(1)10$, numerical comparison of the exact and approximate values of $e^{-\frac{1}{2}t}L_{10}(t)$ for $0.5\leqq t\leqq 34$, and a 4-decimal table of $e^{-\frac{1}{2}t}L_n(t)$ for $n=1(1)10$ and

$$t=0.1(0.1)1(0.25)3(0.5)6(1)14(2)34.$$

A. Erdélyi (Edinburgh).

McLachlan, N. W. Computation of the solution of Mathieu's equation. Philos. Mag. (7) 36, 403–414 (1945). [MF 15478]

The purpose of this paper is to describe methods of computing solutions of Mathieu's equation, other than Mathieu functions of integral order. The author refers to real parameter values for which all solutions are bounded throughout an infinite range of the (real) independent variable. The first procedure dealt with is a modification of a method given by E. L. Ince [Philos. Mag. (7) 6, 547–558 (1928)]. An alternative procedure for the calculation of characteristic values is described, indicated as being simpler in some cases. Upon evaluation of the characteristic values, the Fourier coefficients of the solutions are calculated. Both methods proceed by trial and error and by successive approximations having continued fractions as background. An application of the methods is given to parameter values for which not all solutions are bounded. An appendix contains details of the procedure. *M. J. O. Strutt.*

Vernotte, Pierre. Comment rendre plus sûre la formulation mathématique d'une loi expérimentale. C. R. Acad. Sci. Paris 222, 55–57 (1946). [MF 15980]

Okaya, Tokiharu. Numerical tables of Tchebychef's q-functions and their integrated and derivated functions. Proc. Phys.-Math. Soc. Japan (3) 23, 788–799 (1941). [MF 15009]

The problem of approximating an arbitrary function $f(x)$ for $0<x<n$ by a suitable linear combination

$$F_k(n,x)=a_0+a_1q_1(n,x)+\cdots+a_kq_k(n,x)$$

of k fixed polynomials $q_\nu(n,x)$ was considered by Chebyshev, who showed that the sum of the squares of the discrepancies $\sum_{r=0}^{n-1}\{f(r)-F_k(n,r)\}^2$ is minimized by choosing

$$q_\nu(n,x)=\nu!\,2^{-\nu}\sum_{s=0}^{\nu}\binom{\nu+s}{s}\binom{\nu-n}{\nu-s}\binom{x}{s}$$

and determining the a's by

$$S_\nu(n)a_\nu=\sum_{r=0}^{n-1}f(r)q_\nu(n,r),\qquad S_\nu(n)=\sum_{r=0}^{n-1}\{q_\nu(n,r)\}^2.$$

The present paper gives the exact values of

$$q_\nu(n,r),\quad \frac{d}{dr}q_\nu(n,r),\quad \int_0^r q_\nu(n,x)dx$$

for $r=1(1)n-1$, $\nu=1(1)6$, $n=5(1)12$ and also of $S_\nu(n)$ for the above values of ν and n. *D. H. Lehmer.*

Simonsen, W. On the derivation of interpolation formulas. Mat. Tidsskr. B. **1946**, 135–144 (1946). (Danish) [MF 16315]

Par une voie élémentaire fort simple, sans emploi de notations symboliques, l'auteur établit la plupart des formules d'interpolation usuelles: Stirling, Bessel, Gauss, Everett et Steffensen. *J. Favard* (Paris).

Schoenberg, I. J. Contributions to the problem of approximation of equidistant data by analytic functions. Part A. On the problem of smoothing or graduation. A first class of analytic approximation formulae. Quart. Appl. Math. 4, 45–99 (1946). [MF 15945]

The author first considers a transformation defined by means of a sequence of real numbers $\{L_n\}$ $(L_n=L_{-n})$, which transforms a sequence $\{y_n\}$ into a sequence $\{F_n\}$ by the relation $F_n=\sum_{\nu=-\infty}^{\infty} y_\nu L_{n-\nu}$. Defining the characteristic function $C_z(u)$ of any sequence $\{z_n\}$ as $C(u)=\sum_{n=-\infty}^{\infty} z_n e^{inu}$, it follows that $C_y(u)C_L(u)=C_F(u)$, $C_L(u)=L_0+2\sum_{n=1}^{\infty} L_n \cos nu$, and

$$\sum_{n=-\infty}^{\infty} (\Delta^m F_n)^2=\frac{1}{2\pi}\int_0^{2\pi} \{2 \sin (u/2)\}^{2m}|C_y(u)|^2\{C_L(u)\}^2 du,$$

for $m\geqq 0$. This suggests defining the transformation $\{L_n\}$ as a smoothing formula if $\sum_n L_n=1$, $\sum_n|L_n|<\infty$, and $|C_L(u)|\leqq 1$ for $0\leqq u\leqq 2\pi$. It is shown that a smoothing formula has the property of transforming into themselves all sequences $\{y_n\}$ which may be represented by polynomials in n of degree $2k+1$ or less, if and only if $C_L(u)-1$ has at $u=0$ a zero of order $2k+2$.

An interpolation formula is defined by an even function $L(x)$ of a real variable x, which transforms a sequence $\{y_n\}$ into a function $F(x)=\sum_{n=-\infty}^{\infty} y_n L(x-n)$. Defining the characteristic function of $L(x)$ as $g(u)=\int_{-\infty}^{\infty} L(x)e^{iux}dx$, it is shown that the transformation has the property $F(n)=y_n$ if and only if $\sum_{n=-\infty}^{\infty} g(u+2\pi n)\equiv 1$. Furthermore, a sequence $\{y_n\}$ which may be represented by a polynomial of degree $k-1$ is transformed into that polynomial if and only if $g(u)-1$ has a zero of order k at $u=0$ and $g(u)$ has zeros of order k at $u=2\pi n$ $(n\neq 0)$.

These principles are used to derive a number of particular formulae for interpolating either by means of spliced polynomial arcs or by means of functions analytic over the entire range. Tables are appended to facilitate interpolation by analytic functions. *T. N. E. Greville.*

Krochmal, S. Einige Untersuchungen auf dem Gebiete der Theorie der kleinsten Quadrate. Leningrad State Univ. Annals [Uchenye Zapiski] **83** [Math. Ser. 12], 150–198 (1941). (Russian. German summary) [MF 16494]

The subtitle reads: Ein Kriterium um das Resultat einer Ausgleichungsrechnung zu schätzen. Consider a system of m equations $\sum c_{ik}x_k=L_i$ with n unknowns, $n<m$. Let x_k^0 be the least square solutions and $\epsilon_i=L_i-\sum c_{ik}x_k^0$ their residuals. Without loss of generality it can be assumed that the matrix is normed in such a way that for each column $\sum_i c_{ik}^2=1$. It is assumed that the c_{ik} are random variables such that for every k the probability of finding the point $c_{1k}, c_{2k}, \cdots, c_{mk}$ in a region D of the unit sphere is given by the measure of D; variables of different columns are mutually independent. Under these circumstances the familiar measure of improvement $R=\sum\epsilon_i^2/\sum L_i^2$ becomes itself a random variable assuming values in the interval $(0, 1)$. The author finds the expectation of R to be $E(R)=(m-n)/m$, and the probability density

$$F(x)=B_x(\tfrac{m-n}{2}, \tfrac{n}{2})/B(\tfrac{m-n}{2}, \tfrac{n}{2}),$$

where $B_x(a, b)$ is the incomplete beta function. The proof is based on a gradual building up of the solution in terms of vectors which depend only on one column of the matrix (c_{ik}) at a time. The author concludes that the improvement is significant only if R is essentially smaller than its expectation. Examples taken from astronomy are given. As a corollary the author obtains that the expectation of the determinant of (c_{ik}) is $n!/n^{n+1}$. *W. Feller.*

Frazer, R. A. Bi-variate partial fractions and their applications to flutter and stability problems. Proc. Roy. Soc. London. Ser. A. **185**, 465–484 (1946). [MF 15925]

The author finds a convenient method for constructing interpolation polynomials $P(x, y)$ of degree n in two variables; it may easily be extended to polynomials in several variables. Let $L_i\equiv y-p_ix+q_i$ be a system of straight lines no two of which are parallel and no three of which are concurrent; then, with the aid of $n+2$ such lines, he assumes the expansion

$$\frac{P(x, y)}{L_1L_2\cdots L_{n+2}}=\sum_{i\neq j}\frac{A_{ij}}{L_iL_j},$$

so that the constants A_{ij} are given by

$$A_{ij}=P(x_{ij}, y_{ij})/s_{ij}^{(n+2)},\quad s_{ij}^{(n+2)}=\prod_{k=1}^{n+2} L_k(x_{ij}, y_{ij}),\quad k\neq i\neq j.$$

Here (x_{ij}, y_{ij}) is the point of intersection of L_i and L_j. Alternatively, with the aid of $n+1$ such straight lines L_k, he writes

$$\frac{P(x, y)}{L_1\cdots L_{n+1}}=\sum_{i\neq j}\frac{A_{ij}}{L_iL_j}+\sum_{i=1}^{n+1}\frac{D_i}{L_i}.$$

Here the A_{ij} are calculated as above, with $n+2$ replaced by $n+1$. Furthermore, $D_i=P_0(1, p_i)/d_i^{(n+1)}$, where $d_i^{(n+1)}=\prod_{k=1}^{n+1}(p_i-p_k)$ and where $P_0(x, y)$ are the terms of $P(x, y)$ of order n.

For practical use the L_i and thus the interpolation points may be chosen once for all. The author chooses the tangents of a parabola and gives tables for their points of intersection (x_{ij}, y_{ij}) and for the $s_{ij}^{(n+1)}$ and $d_i^{(n+1)}$ for $n=3, 4, \cdots, 9$. Similarly, for polynomials in three variables one may write

$$\frac{P(x, y, z)}{L_1L_2\cdots L_{n+3}}=\sum_{i\neq j\neq k}\frac{A_{ijk}}{L_iL_jL_k}.$$

This method is applied to the expansion of the Lagrangian stability determinants of the restricted type,

$$\Delta(\lambda, y)=|a_{ij}\lambda^2+b_{ij}\lambda+c_{ij}+e_{ij}y|,$$

and to the solution of their critical equations, and furthermore to the solution of the characteristic equation of a determinant. *E. Bodewig* (The Hague).

Heinrich, Helmut. Zur rechnerischen Auflösung einer Gleichung vierten Grades. Z. Angew. Math. Mech. **21**, 304–307 (1941). [MF 15863]

A compact method for simultaneous computation of all (real and complex) roots of biquadratic and quintic equations with real coefficients, based on the fact that for $f(x)=x^4+ax^2+bx+c=0$ the resolvent cubic $R(z)=z^3+\alpha z^2+\beta z+\gamma=0$, $\alpha=2a$, $\beta=a^2-4c$, $\gamma=-b^2$, necessarily has a positive root. Let z be such a root, preferably the greatest, if there are more than one. Then $y=\pm\frac{1}{2}\sqrt{z}$, $f'(y)=y(z+2a)+b$, and $x=y\pm\sqrt{(-f'(y)/(4y))}$ gives the

four roots of $f(x)=0$. The reviewer does not claim detailed knowledge of the very extended literature of this field, but believes that the paper offers a practical and economical method when all four roots are required. A schedule for computation and a numerical example are given.

A. J. Kempner (Boulder, Colo.).

Massera, Jose L. The method of Graeffe for solving algebraic equations. Bol. Fac. Ingen. Montevideo 3 (Año 10), 1–20 (1945). (Spanish) [MF 16165]

Hamilton, Hugh J. Roots of equations by functional iteration. Duke Math. J. 13, 113–121 (1946). [MF 15880]

The author studies the convergence of iterative approximations to a solution of $\varphi(z)=0$. In the case where $z_n=f(z_n-1)$, $f(z)=\varphi(z)+z$, and z lies in a metric space, he proves several theorems and connects his results with those of Hildebrandt and Graves [Trans. Amer. Math. Soc. 29, 127–153 (1927)]. In the case where z is a real or complex number, he considers the introduction of other forms of f, for example

$$f(z)=z-6\varphi(2\varphi'^2-\varphi\varphi'')/(6\varphi'^3+\varphi^2\varphi'''-6\varphi\varphi'\varphi'')+a\varphi^3/(6\varphi'^3+\varphi^2\varphi''-6\varphi\varphi'\varphi'')$$

and obtains results of specific and general application as to when the convergence is faster than, equal to or slower than that of a geometric series of ratio r. Some of these results are closely related to those of P. A. Samuelson [J. Math. Phys. Mass. Inst. Tech. 24, 131–134 (1945); these Rev. 7, 337]. *J. W. Tukey* (Princeton, N. J.).

Kamela, Czeslaw. Die Lösung der Normalgleichungen nach der Methode von Prof. Dr. T. Banachiewicz (sogenannte "Krakovianenmethode"). Schweiz. Z. Vermessgswes. Kulturtech. 41, 225–232, 265–275 (1943). [MF 15574]

This paper gives first an outline of the method of solving normal equations by means of "Cracovians" [Banachiewicz, Astr. J. 50, 38–41 (1942); these Rev. 4, 90] and illustrates the procedure by means of a numerical example. An interesting feature of the paper is an analysis of the number of operations of multiplication, division, and addition or subtraction required to solve n normal equations in n unknowns, first, for Gauss's method and, second, by the Cracovian method. The results are exhibited in tabular form, of which the following is a sample for $n=9$.

	×	÷	+, −	Total
G:	285	54	285	624
C:	285	45	285	615

W. E. Milne (Corvallis, Ore.).

Jensen, Henry. An attempt at a systematic classification of some methods for the solution of normal equations. Geodætisk Institut, København, Meddelelse no. 18, 45 pp. (1944). [MF 16181]

Explaining and comparing the usual methods for a direct solution (that is, noniterative) of linear equations, the author arrives at the following conclusions. The determinantal solution by Chiò's rule and the method of equal coefficients are only practicable in the case of few equations. Gauss's method gives a very plain solution for all systems. Banachiewicz's method of Cracovians is not superior to the Gaussian algorithm. Boltz's and Krüger's methods are useful in special systems, while the best of all methods seems to be Cholesky's. The explanations are illustrated by two examples which are computed by all methods or a part of them. *E. Bodewig* (The Hague).

Berry, Clifford E., Wilcox, Doyle E., Rock, Sibyl M., and Washburn, H. W. A computer for solving linear simultaneous equations. J. Appl. Phys. 17, 262–272 (1946). [MF 15901]

This computer, like its predecessors by Mallock and by Wilbur, is for solving simultaneous equations with 3- or 4-figure coefficients. Whereas earlier machines have employed direct solution, this one is an adaptation of iterative methods. Analogous electrical circuits are developed, leading to an instrument in which only small readings are required, except for the final values of the unknowns. A commercial model has been produced for any number of equations up to twelve.

Naturally the accuracy obtainable depends on whether the equations are well-conditioned or otherwise. In favourable circumstances a strong 3-figure or weak 4-figure accuracy is obtained. It is claimed that the time taken is of the order of one fifth of that required by conventional methods with desk calculators. Although the computer was developed for equations arising in mass spectrometry, especially with oil mixtures, it should find many applications in statistics, engineering and aircraft design.

L. J. Comrie (London).

Saunderson, J. L., and Grossman, H. H. Simultaneous linear equations in absorption spectrophotometry and mass spectrometry. J. Opt. Soc. Amer. 36, 243 (1946). [MF 15947]

This letter was inspired by an article by L. J. Comrie dealing with the solution of simultaneous linear equations, which pointed out a technique for lessening the numerical labour when only the right-hand side varies in several sets of equations. It is shown how this may be applied to the electrical "Spectro-computer" designed by Morgan and Crawford and manufactured by Engineering Laboratories, Inc. *L. J. Comrie* (London).

Federhofer, Karl. Über besondere Seilkurven. Ein Beitrag zur graphischen Analysis. Z. Angew. Math. Mech. 21, 233–241 (1941). [MF 15855]

The author develops a method for solving by graphical means an ordinary differential equation involving polar coordinates. He illustrates it for the case of a flexible string under equilibrium when subjected to a central force proportional to a power of the polar distance. *P. Franklin.*

Nevanlinna, Rolf. Berechnung der Normalflugbahn eines Geschosses. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. 15, 8 pp. (1943). [MF 16501]

The author proposes a method "which in comparison to most of the usual methods possesses simplicity and speed." The numerical method as briefly sketched seems identical in all details to that used since 1917 by most computers in the Ordnance Department of the U. S. Army and generally called "Moulton's method." It is explained in many texts prepared by the Army and in F. R. Moulton's "New Methods in Exterior Ballistics" [University of Chicago Press, 1926]. *A. A. Bennett* (Providence, R. I.).

Sugar, Alvin C. On the numerical treatment of forced oscillations. Quart. Appl. Math. 4, 193–196 (1946). [MF 16965]

Musson-Genon, René. Résolution de certaines équations aux dérivées partielles au moyen de la cuve électrolytique. C. R. Acad. Sci. Paris 222, 274–275 (1946). [MF 16001]

Salzer, Herbert E. Note on coefficients for numerical integration with differences. J. Math. Phys. Mass. Inst. Tech. 25, 86–88 (1946). [MF 16152]

In an earlier paper [same J. 24, 1–21 (1945); these Rev. 7, 85] Lowan and the author published a table of the integrals of Lagrangian coefficients for obtaining the integral of a function at intermediate points in terms of given equally spaced values of the function without the use of differences. The present paper provides a list of formulas by means of which the tables of the earlier paper can be conveniently used for the integration of the Gregory-Newton, Newton-Gauss (forward and backward), Everett, and Steffensen interpolation formulas. *W. E. Milne* (Corvallis, Ore.).

Lösch, Friedrich. Zur praktischen Berechnung der Eigenwerte linearer Integralgleichungen. Z. Angew. Math. Mech. 24, 35–41 (1944). [MF 15847]

The problem of determining approximately the first few characteristic numbers of the linear integral equation

$$y(x)-\lambda\int_a^b K(x,s)y(s)ds=0$$

is reduced to an algebraic problem by the familiar procedure of setting

$$K(x,s)=\sum_{i=1}^n\varphi(x)\psi_i(s),\quad y(x)=\sum_{i=1}^n a_i\psi_i(x),$$

where the ψ's form an appropriate set of orthogonal and normalized functions in the interval (a, b) and the a's are coefficients to be determined. The desired values of λ are roots of the nth degree equation

$$\begin{vmatrix} 1-\lambda K_{11} & \cdots & -\lambda K_{1n} \\ -\lambda K_{n1} & \cdots & 1-\lambda K_{nn} \end{vmatrix}=0,$$

in which

$$K_{ij}=\int_a^b\psi_i(x)\int_a^b K(x,s)\psi_j(s)dsdx.$$

The method is illustrated by finding the three lowest frequencies of vibration of a beam fixed at one end and free at the other and having variable cross section and variable moment of inertia of cross section. For this problem the ψ's are taken as orthogonal polynomials satisfying the boundary conditions. *W. E. Milne* (Corvallis, Ore.).

Chandrasekhar, S. On the radiative equilibrium of a stellar atmosphere. IX. Astrophys. J. 103, 165–192 (1946). [MF 16293]

[For part VIII see the same J. 101, 348–355 (1945); these Rev. 6, 244.] The problem of the diffuse radiation from a semi-infinite plane-parallel atmosphere illuminated by a beam of incident radiation is attacked by the method of replacing the integrals occurring in the solution of the equation of the transfer of radiation by sums according to Gauss's formula for numerical quadratures. Two cases are dealt with, namely: case (i) in which the phase-function, giving the probability that a pencil of radiation will be scattered at an angle Θ to the incident beam, is $\lambda(1+x\cos\Theta)$, $0\leqq\lambda\leqq1$, $-1\leqq x\leqq1$; case (ii) in which the phase-function is $1+\cos^2\Theta$. Simple closed expressions for the angular distribution of the reflected radiation in the nth approximation are obtained. These can be reduced to numerical solutions by solving for the characteristic roots of certain algebraic equations. A number of constants and functions required for the computations up to the third approximation in case (i) and up to the fourth in case (ii) are tabulated.

G. C. McVittie (London).

Kogbetliantz, Ervand George. Estimating depth and excess-mass of point-sources and horizontal line-sources in gravity prospecting. Geophysics 11, 195–210 (1946). [MF 16325]

Nikolayeff, P. L'anamorphose des équations. Rec. Math. [Mat. Sbornik] N.S. 17(59), 253–266 (1945). (Russian. French summary) [MF 16670]

[Cf. C. R. (Doklady) Acad. Sci. URSS (N.S.) 28, 582–584, 774–777 (1940); 47, 82–86, 155–158 (1945); Uchenye Zapiski Moskov. Gos. Univ. Matematika 73, 83–98, 99–116, 117–128 (1944); these Rev. 2, 240; 7, 221.]

L'objet de la présente communication est un des problèmes fondamentaux de la nomographie: le problème de trouver le critère pour qu'une équation algébrique soit susceptible d'anamorphose et de donner un algorithme pour réduire le polynome de cette équation au déterminant de Masseau (pourvu que c'est possible). Les articles précédents de l'auteur donnent la solution de ce problème dans le cas où l'équation considérée est susceptible d'anamorphose immédiate. L'article présent contient la solution du problème à l'aide du facteur anamorphosant.

Les trois premiers théorèmes donnent les procédés de trouver l'équation d'involution pour chacune des variables, de normaliser l'équation et de dégager la part normale de l'équation, qui est l'équation normale de Masseau dans le cas où l'équation donnée est susceptible d'anamorphose. Le problème de construire l'anamorphose d'une équation est ainsi réduit au cas de l'équation normale. Le théorème 4 donne une méthode de symétrisation de l'équation, ce qui est toujours possible, pourvu que l'équation considérée admet un facteur anamorphosant.

La part suivante de l'article contient la solution du problème d'anamorphose des équations symétriques normalisées. Enfin nous donnons quelques procédés pratiques de construire l'anamorphose d'une équation algébrique quelconque (à trois variables), ainsi que des critères pour qu'une équation soit susceptible d'anamorphose.

Author's summary.

Gruner, W. Über eine Ungleichung und ihre Anwendung bei der Abschätzung des Deckungskapitals einer gemischten prämienpflichtigen Versicherung mit steigender Todesfallsumme. Mitt. Verein. Schweiz. Versich.-Math. 45, 385–403 (1945). [MF 15426]

Let $f(t)$ be a continuous, positive, nonincreasing function, defined for $0\leqq t\leqq n$, such that $\log f(t)$ is a concave function of t. The author proves the inequality

$$\int_t^n\tau f(\tau)d\tau\int_0^t f(\tau)d\tau-\int_t^n f(\tau)d\tau\int_0^t\tau f(\tau)d\tau\leqq\tfrac{1}{8}n^2f(t)\int_0^n f(\tau)d\tau$$

for $0\leqq t\leqq n$; the equality sign is true if and only if $f(t)$ is constant and $t=\frac{1}{2}n$. This inequality is then used to estimate the premium reserve W_t of an insurance policy under which the amount 1 becomes payable if the assured is alive after n years, and the amount τ/n if he dies after τ years. The estimate obtained is $t/n-W_t\leqq\frac{1}{8}\delta n$, where δ is the force of interest. The only assumption on mortality used in obtaining this estimate is that the force of mortality is a nondecreasing function of age. *Z. W. Birnbaum.*

Bonferroni, C. E. La condizione d'equilibrio per operazioni finanziarie finite od infinite. Giorn. Ist. Ital. Attuari 11, 190–213 (1940). [MF 16621]

The principle of equivalence regulates the relation between payments and premium in insurance transacted by private companies. In the field of social insurance this principle cannot be applied without modification. This is due to the fact that social insurance is an infinite financial operation, whereas actuarial mathematics considers only finite operations as a rule. The author discusses quite general financial operations, which may be finite or infinite, admitting even nonmultiplicative laws of interest. He formulates a condition of equilibrium, replacing the principle of equivalence for these operations. *E. Lukacs.*

Zwinggi, Ernst. Über die Bedeutung der infinitesimalen Betrachtungsweise für die Grundlagen der Versicherungstechnik. Mitt. Verein. Schweiz. Versich.-Math. 46, 89–104 (1946). [MF 16687]

MECHANICS

Landsberg, Peter T. On plane rotations in *n* dimensions. J. Appl. Phys. 17, 60–61 (1946). [MF 15105]

A generalization to n dimensions of an operational equation derived by E. T. Benedikt [same J. 15, 613–615 (1944); 16, 551 (1945); these Rev. 6, 23; 7, 90] for the representation of rigid rotations. The author uses the concepts and nomenclature of modern algebra instead of the elementary but relatively awkward formulas attributed to Euler and used by Benedikt. The author himself points out that his result is essentially well known. *D. C. Lewis.*

Gasparini, Ida. Sulla composizione di spostamenti rigidi secondo Poincaré. Boll. Un. Mat. Ital. (2) 4, 31–37 (1942). [MF 16054]

It is proved that the resultant displacement, in the sense of Poincaré, of two rigid nontranslatory displacements is itself rigid only if the two component displacements are rotations about the same axis in opposite senses and through supplementary angles. *D. C. Lewis* (College Park, Md.).

Palacios, Julio. Mouvements d'un solide soumis à l'action d'un couple de direction fixe dans l'espace. Ann. Fac. Sci. Univ. Toulouse (4) 7, 123–131 (1945). [MF 16179]

An investigation of the motion of a solid of revolution acted upon by a constant couple. In general there is asymptotic approach to a motion of uniformly accelerated precession (with vanishing oscillations of nutation) of the axis of symmetry about the axis of the couple. If the initial moment of momentum is zero, these asymptotic conditions are fulfilled at the outset and precise information can also be given concerning the motion of the body about its axis of symmetry. There is also brief consideration of the general rigid body acted upon by a couple of constant direction (but nonconstant magnitude) in the special case when the initial moment of momentum has the same direction as the couple. *D. C. Lewis* (College Park, Md.).

Palacios, J. Mouvement d'un solide mis en rotation par l'intermédiaire d'un joint élastique. Ann. Fac. Sci. Univ. Toulouse (4) 7, 132–138 (1945). [MF 16180]

A theoretical explanation of the fact that the motion of a rigid body suspended at the end of a rapidly twisting string asymptotically approaches a steady and stable rotation about its principal axis of inertia. *D. C. Lewis.*

Kohn, Walter. Contour integration in the theory of the spherical pendulum and the heavy symmetrical top. Trans. Amer. Math. Soc. 59, 107–131 (1946). [MF 15319]

This paper primarily concerns the motion of a gyroscope under the influence of gravity alone, the gyroscope being supported at a fixed point of its axis not at the center of gravity. The motion of a spherical pendulum may be regarded as the motion of such a gyroscope when β, the component of angular momentum about the spin axis, is zero. Although the author's results on the gyroscope are thus immediately applicable to the spherical pendulum, the latter is given a separate treatment in order to introduce the reader as simply as possible to the essentials of the method.

In the sequel Φ denotes the advance of azimuth as the center of gravity passes from its lowest level z_1 to its highest level z_2, the zero level being taken at the point of support. In the case of the pendulum and the retrogradely precessing gyroscope, $z_1+z_2<0$. Furthermore, Φ_0 denotes that part of the advance of the azimuth from z_1 to z_2 which takes place in the lower hemisphere, so that $\Phi_0<\Phi$ or $\Phi_0=\Phi$ according as $z_2>0$ or $z_2\leqq 0$. The component of angular momentum about the vertical direction is denoted by α.

For the spherical pendulum and the retrogradely precessing gyroscope in which $|\alpha|>|\beta|$, the inequalities $\frac{1}{2}\pi<\Phi_0\leqq\Phi<\pi$ are established. For the spherical pendulum the inequalities $\frac{1}{2}\pi<\Phi<\pi$ are due to Puiseux and Halphen, respectively. The author's essential contribution is, therefore, the introduction of the quantity Φ_0 in addition to the generalization to the case of the gyroscope. The author has also systematically investigated the other gyroscopic cases. The lower and upper boundaries of Φ depend very heavily upon the ratio α/β. They present especially peculiar behavior when $\alpha/\beta=\pm 1$. In any case, it is shown that, for a fixed α/β, Φ may take on any value between its lower and upper boundaries.

The method used is that of contour integration. This method, which was first introduced by Hadamard in this connection, is appropriate because Φ is given explicitly as a definite elliptic integral. The author has studied a greater variety of contours than have previously been considered and has also introduced other devices, such as the concept of "conjugate motions" and the use of infinite sequences of motions in order to prove that the established inequalities cannot be improved. *D. C. Lewis.*

Udeschini, Paolo. Giroscopio rotolante sopra un piano orizzontale in presenza di un polo magnetico. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. 77, 384–398 (1942). [MF 16255]

This discussion of the motion of a heavy top spinning on a horizontal plane is characterized by the presence of forces in addition to those of gravity and the reaction of the horizontal plane. These forces are assumed to consist of a couple with respect to the center of gravity as well as a horizontal force applied at this same point and passing through a fixed vertical line. Certain small quantities are assumed negligible. Special attention is devoted to cases in which the axis of the top makes a constant angle with the vertical. In such

cases the top has uniform circular motion about the fixed vertical line. *D. C. Lewis* (College Park, Md.).

Četajev, N. G. Stability of rotatory motion of a projectile. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] 10, 135–138 (1946). (Russian. English summary) [MF 16840]

Pugachev, V. S. Approximate method of investigation of plane non-linear oscillations of a projectile with a stabilizer. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] 10, 139–152 (1946). (Russian. English summary) [MF 16841]

Galin, L. A. Determination of the differential equation of an instrument on the basis of experimental results in constrained oscillation tests. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] 10, 93–100 (1946). (Russian. English summary) [MF 16838]

The following problem is solved. In a certain mechanical system, whose motion is described by linear differential equations with constant coefficients, constrained oscillations arising through a disturbance x_0 varying according to the sinusoidal law result in changes in the value of x_1 according to the sinusoidal law likewise. Let the value of the amplitude $B(\omega)$ and of the phase shift $\varphi(\omega)$ be known for corresponding frequencies of ω. The relationship between x_1 and x_0 is given by the formula

$$\left[\prod_{k=1}^{n}(d/dt-q_k)^{\beta_k}\right]x_1=\left[C\prod_{k=1}^{m}(d/dt-p_k)^{\alpha_k}\right]x_0,$$

where q_k are the poles, p_k are zeros with orders and powers β_k and α_k, respectively; n is the number of poles and m is the number of zeros resulting from an analytic continuation of the function

$$\int_{-i\infty}^{i\infty} A^{-1}B(-i\zeta)e^{i\varphi(-i\zeta)}(\zeta-z)^{-1}d\zeta$$

determined for the right semiplane throughout the entire plane; C is the coefficient of the term with highest power of z in the expansion of this function in the vicinity of the infinite point. *From the author's summary.*

Vlach, Bohumil. Die Zentralbewegungen mit der dem Quadrate der Entfernung umgekehrt proportionalen Beschleunigung und das Prinzip der kleinsten Aktion. Věstník Královské České Společnosti Nauk. Třída Matemat.-Přírodověd. 1944, 33 pp. (1944). (Czech. German summary) [MF 16117]

Thomas, T. Y. Reducible dynamical systems. J. Math. Phys. Mass. Inst. Tech. 25, 89–91 (1946). [MF 16153]

A conservative holonomic dynamical system with n degrees of freedom, with kinetic energy T, and with potential energy U, is said to be reducible if it is possible to introduce coordinates in such a manner that $T=T_1+T_2$ and $U=U_1+U_2$, where U_1 and T_1 depend only upon $x_1, \cdots, x_r$ and their derivatives while U_2 and T_2 depend only upon $x_{r+1}, \cdots, x_n$ and their derivatives. A theorem is established giving a necessary and sufficient condition for reducibility in terms of the existence of an integral quadratic in the velocities (distinct from the energy integral), the covariant derivative of whose coefficient tensor vanishes. The proof involves the theory of parallel vector spaces in a Riemann space. *D. C. Lewis* (College Park, Md.).

Faure, Robert. Intégration des équations; cas de Liouville. C. R. Acad. Sci. Paris 222, 1032–1033 (1946). [MF 16380]

It is shown by means of an example that a dynamical system which admits separation of the variables in the classical theory does not necessarily admit separation of the variables in the quantum theory. *D. C. Lewis.*

Hagihara, Yusuke. On the osculating representation for a dynamical system with slow variation. I. Proc. Imp. Acad. Tokyo 20, 617–621 (1944). [MF 14932]

The system

$$dx_i/dt=\partial H/\partial y_i,\quad dy_i/dt=-\partial H/\partial x_i,\quad i=1, 2, \cdots, m,$$
$$d\xi_j/dt=\partial H/\partial \eta_j,\quad d\eta_j/dt=-\partial H/\partial \xi_j,\quad j=1, 2, \cdots, n,$$

is assumed to have a solution $x_i=y_i=0$, $\xi_j=A_j$, $\eta_j=B_j$, where A_j and B_j are arbitrary constants. Here H is analytic in x_i, y_i and t, periodic in t, and has first and second partial derivatives which satisfy Lipschitz conditions in ξ_j and η_j. It is also assumed that H satisfies further conditions roughly described by saying that the above solution is of formal stable type.

By use of the familiar formal series of dynamics, the author normalizes H up to the terms of degree s in x_i and y_i. The coefficients in H of terms higher than the second are in general functions of ξ_j and η_j. Under certain conditions these normalized equations are readily seen to have "approximate" solutions of quasiperiodic type. The object of this paper is to formulate as precisely as possible the sense of this approximation. It is not practicable to review the complicated details here. The paper itself is hardly more than an abstract in which the proofs are indicated by references to preceding work of the author and to Bohl, Esclangon and Kronecker on quasiperiodic functions and Diophantine approximation. *D. C. Lewis* (College Park, Md.).

Hagihara, Yusuke. On the osculating representation for a dynamical system with slow variation. II. Proc. Imp. Acad. Tokyo 20, 622–626 (1944). [MF 14933]

This paper treats the same Hamiltonian system as the paper reviewed above under the additional assumption that H is analytic in ξ_j and η_j. The normalization is then carried out with respect to ξ_j and η_j as well as with respect to x_i and y_i. The "curtailed" system is then certain to have a quasiperiodic solution in which the periods for ξ_j and η_j are long compared with those for x_i and y_i. The approximation of this quasiperiodic solution of the curtailed system to the solution of the exact system is discussed. *D. C. Lewis.*

Schönberg, Mario, and Camargo Schützer, Walter. Conditions for the existence of a potential. I. Anais Acad. Brasil. Ci. 17, 167–173 (1945). (Portuguese) [MF 14548]

Schönberg, Mario, and Camargo Schützer, Walter. Conditions for the existence of a potential. II. Anais Acad. Brasil. Ci. 17, 175–179 (1945). (Portuguese) [MF 14549]

The theory which forms the basis for both papers differs from Newtonian mechanics in that a single particle is characterized by a spin vector in addition to its position and mass; the differential equations of motion include, in addition to those connecting acceleration and mass, the equation $S=M$, where S is the spin vector and M the momentum vector. The situation is considered when the force and momentum vectors can be derived from a potential by

equations of the types

$$F_s = -\partial V/\partial x, \quad M_s = S_y\partial V/\partial S_z - S_z\partial V/\partial S_y.$$

The discussion is devoted to the derivation of the integrability conditions of these relations considered as equations for V.

In the first paper the potential is assumed to depend only on the position and the spin; the cases of a single particle and of a system of particles are considered. In the second paper the force and the spin vectors depend also on the velocities of the particles, and the particles are subjected to holonomic constraints, that is, the coordinates of the particles (but not the spin components) are assumed to satisfy finite equations. The results of this paper may be considered as extensions of the results of Helmholtz and Koenigsberger. *G. Y. Rainich* (Ann Arbor, Mich.).

Okhotsimsky, D. E. On the theory of rocket propulsion. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] 10, 251–272 (1946). (Russian. English summary) [MF 16848]

Ackeret, J. Zur Theorie der Raketen. Helvetica Phys. Acta 19, 103–112 (1946).

The author reviews the theory of rockets, including relativistic effects. He then considers the possibility of an atomic powered rocket reaching the moon or a neighboring planet. He finds that there is sufficient energy, but considers that the main difficulty will be the generation of the necessary high velocities without excessively high temperatures.
P. Franklin (Cambridge, Mass.).

Kochin, N. E. Form taken by the cable of a fixed barrage balloon under the action of wind. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] 10, 153–164 (1946). (Russian. English summary) [MF 16842]

Astronomy

Polak, J. F. Diagramms for determination the elements of an orbit in the two-body problem from the velocity at a given point. Astr. J. Soviet Union [Astr. Zhurnal] 21, 99–110 (1944). (Russian. English summary) [MF 15905]

The purpose of this paper is to facilitate the determination of the orbital elements of a meteor or a comet subject only to the force of gravity of the sun. The author assumes that the observations have given the velocities x, y in km./sec. along the radius vector and at right angles to it. If c denotes one of the elements of the orbit, other than the inclination or the ascending node, then the expression $f(x, y, c) = 0$ defines a family of curves of equal values of c. These curves have been reproduced for the true anomalies, the eccentricities, the perihelion distances and the intervals of time elapsed since perihelion passage. The diagrams have been constructed for a distance from the sun of one astronomical unit, but instructions are given for converting the results to an arbitrary value of R. The range in x and y is from 0 to 70 km./sec.; elliptical as well as hyperbolical meteors are thus included. The author points out that the solutions for the semimajor axis and the parameter of the orbit are trivial: the former is represented by concentric circles around the origin, the latter by a series of straight lines which are parallel to the x-axis. *O. Struve.*

Polak, J. F. Diagrams for solution of the problem of two bodies from the velocity vector in the case of repulsion. Astr. J. Soviet Union [Astr. Zhurnal] 22, 283–292 (1945). (Russian. English summary) [MF 15437]

The author has prepared four diagrams showing, respectively, four families of curves of equal values of v (true anomaly), e (eccentricity), q (perihelion distance) and T (time elapsed since perihelion distance). In each diagram the abscissa x is the radial velocity-component of the particle at distance $R=1$ astronomical unit, and the ordinate y is the corresponding transverse velocity-component. Both x and y are expressed in astronomical units per $1/k = 58.13$ days. The acceleration of repulsion is taken as $k^2\mu = 1$ and the force is assumed to vary inversely as the square of the distance. If c is one of the four elements, then the isocurves are computed from expressions of the form $f(x, y, c) = 0$. The diagrams range in x from 0 to $+3.0$ and in y from 0 to $+2.5$. The lines of equal values of v are hyperbolas whose vertices lie on a curve given by $x^2 = y^2(1+y^2)/(1-y^2)$. The corresponding range in v is from 0° to about 55°. Similarly, the eccentricities range from 1.0 to 9.0, the perihelion distances from 0.2 to 1.0 and the values of T from 0 to 59.6, in units of $1/k$ days. Instructions are given for appropriate changes in the units when $R \neq 1$ and $k^2\mu \neq 1$. The diagrams as printed are about $3\times 2\frac{1}{2}$ inches in size, so that only rough values of the elements can be read off by interpolation between adjoining curves. For example, for v a precision of about 1° should be easily obtainable.

The interest of the diagrams lies in their application to such phenomena as comet tails, where the instantaneous velocities of gaseous condensations subjected to radiation pressure are measured on direct photographs and on radial-velocity spectrograms; condensations in novae; and other cases of expanding atmospheres. These applications are not discussed. *O. Struve* (Williams Bay, Wis.).

✶Vidal Abascal, Enrique. El Problema de la Orbita Aparente en las Estrellas Dobles Visuales. [The Problem of the Apparent Orbit for Visual Double Stars]. Thesis, University of Madrid. Publicaciones del Observatorio de Santiago, II. Consejo Superior de Investigaciones Cientificas, Instituto Nacional de Geofisica, no. 6. Santiago de Compostela, 1944. xv+62 pp. (Spanish. French, English and German summaries)

Numerous methods have been developed for deriving and improving orbits of binary systems in which a large arc of the orbit has been covered by the observations. In the great majority of the known systems, however, the arc covered is comparatively short, the determination of the apparent ellipse is quite uncertain and many trials are necessary to obtain satisfactory results. To shorten the process various graphical methods have been devised. All start with a plot of the observations, through which is drawn an ellipse from which may be derived preliminary values of the period P, time of periastron T, eccentricity e and the ratio b/a of the true semimajor and semiminor axes. These elements are manipulated until the values which best represent the observations are found. In the more commonly used methods the next step is the construction of an auxiliary ellipse by increasing the projection of the minor axis in the ratio of the true axes to give the projection of a circle circumscribed about the major axis. The other elements of the true orbit are then determined from this projection.

The author seeks to simplify the construction of the apparent ellipse and to do away with the auxiliary one by

establishing direct relations between the apparent ellipse and a circle circumscribed about its major axis. These relations are

$$(1)\qquad \tan \varphi = \tan (p-\alpha)\cdot(a/b),$$
$$(2)\qquad \delta = \varphi - \psi,$$
$$(3)\qquad \tan (p-\alpha) = (b/a) \tan (\delta+\psi),$$

in which p is the measured position angle, α is the angle between the X-axis and the major axis of the apparent ellipse, ψ is the angle on the circumscribed circle between the X-axis and periastron, δ is the angle on the circumscribed circle between periastron and a point corresponding to the time of observation. The area A of an eccentric circular sector is evidently a function of δ and e, which determines the origin of the vector defining δ. For a circle of unit radius, values of A corresponding to different values of δ and e may be tabulated. Since A is determined independently from the relation $A=(t-T)\cdot\pi/P$, one may enter the table with A and e and determine δ.

The usual procedure would then be to compute values of p by (3) and compare them with the observed values, adjusting the elements to make $|\Delta p|^2$ a minimum. The author, however, avoids this computation by constructing a bundle of rays representing the values of δ derived from the table and superposing them upon a similar bundle of rays representing the observed position angles in such a manner that the two sets of rays intersect. Since the centers of the two systems are the same, the intersections should, if the observations were perfect and the preliminary elements were correct, lie upon a straight line. The distribution of the intersections of the homologous rays with respect to what the author calls the axis of "perspectivity" gives an idea of both the magnitude and distribution of the errors and forms the basis for judgment for improvement of the orbit. The method is useful primarily in the study of position angles, which in general give better results for short observed arcs, but a formula for treating the distances is given. The method has the virtue of simplicity and avoids the use of a great deal of computation in the preliminary stages of finding an orbit. It is applied to the computation of the orbit of ADS 6871.

Two appendices reviewing several of the methods of binary orbit computation are valuable for the reason that they bring together so much information from scattered sources. The final section contains the table of values of the areas of the eccentric circular sectors for a circle of unit radius for each degree in δ and for values of e in tenths from 0.1 to 0.9. *R. E. Wilson* (Pasadena, Calif.).

Gratton, Livio. Circolazione interna e instabilità nelle binarie strette. Mem. Soc. Astr. Ital. (N.S.) 17, 5–27, 139 (1945). (Italian. Latin summary) [MF 16982]

Il teorema di von Zeipel, generalizzato da S. Chandrasekhar, impone alla legge che governa la produzione di energia in una stella rotante o deformata per effetto di marea una restrizione incompatibile con i fenomeni fisici che sono la causa di tale produzione. La conseguenza è che entro una stella, in cui la distribuzione della densità non è sfericamente simmetrica, si debbono stabilire correnti circolatorie. La grandezza di queste correnti si può calcolare mediante una formula molto generale che viene qui determinata. L'applicazione di tale formula al caso del sole mostra che la circolazione che cosi si ottiene è in una scala assai minore di quella postulata nella teoria di Bjerknes delle macchie solari; essa potrebbe però essere il meccanismo di avviamento della circolazione solare. Applicata al caso di una binaria con le componenti a contatto, la formula generale conduce a risultati più interessanti; nel caso particolare della componente principale di β Lyrae si trova che la circolazione è cosi forte da produrre una fuga di materia nel piano equatoriale in direzione della componente secondaria, conformemente a quanto viene ammesso nella teoria di Kuiper del sistema di β Lyrae. *Author's summary.*

Garcia, Godofredo. Cardinal canonical form of the equations of motion of three bodies with the intervention of dissipative and gyroscopic forces in addition to universal gravitation. Actas Acad. Ci. Lima 9, 29–41 (1946). (Spanish) [MF 16871]

Continuation of two papers in the same Actas 7, 351–360, 361–367 (1944); 8, 3–6 (1945); cf. these Rev. 6, 75, 190.

Pedersen, Peder. Die Librationsellipsen um die Dreieckslibrationspunkte im allgemeinen Dreikörperproblem. Danske Vid. Selsk. Math.-Fys. Medd. 19, no. 7, 25 pp. (1941). [MF 15392]

The author determines the ratios of minor and major axes, orientation of axes and other properties of the elliptic periodic orbits about the libration points of the Lagrangian triangular solution of the three-body problem.
W. Kaplan (Ann Arbor, Mich.).

Pedersen, Peder. Librationspunkte im restringierten Vierkörperproblem. Danske Vid. Selsk. Math.-Fys. 21, no. 6, 80 pp. (1944). [MF 15402]

The author considers the restricted four-body problem for which three finite masses rotate in Lagrangian triangular orbits, while an infinitesimal mass moves in the same plane. The equations for libration are determined; it is shown that, for each given libration position, there is a unique ratio of masses, determined by linear equations. The dependence of the position of the center of mass of the finite bodies on the libration position is examined in great detail, and a number of cases are explored numerically.
W. Kaplan (Ann Arbor, Mich.).

Buchanan, Daniel. A six-body problem. Trans. Roy. Soc. Canada. Sect. III. (3) 39, 1–20 (1945). [MF 15518]

The author determines the periodic and asymptotic orbits near the libration points of a six-body problem in which five equal finite masses form a regular pentagon rotating about the center of mass. [Cf. J. L. Hinrichsen, Amer. Math. Monthly 50, 231–237 (1943); D. Buchanan, Canadian J. Research. Sect. A. 22, 1–25 (1944); these Rev. 4, 227; 5, 191.] *W. Kaplan* (Ann Arbor, Mich.).

Hagihara, Yusuke. On the reducibility of the differential equations in the n-body problem. Proc. Imp. Acad. Tokyo 20, 501–504 (1944). [MF 14915]

Lie's theory of function groups is applied to the reduction of the order of the differential equations of the general n-body problem, and also of the planar case.
W. Kaplan (Ann Arbor, Mich.).

Seares, Frederick H. Regression lines and the functional relation. II. Charlier's formulae for a moving cluster. Astrophys. J. 102, 366–376 (1945). [MF 15112]

[For part I, see the same J. 100, 255–263 (1944); these Rev. 6, 91.] Charlier's formulae for the distance of a moving star cluster lead to a systematic error, as ordinarily used. The error, which arises from hidden regression effects, amounts to about 7 per cent in the case of the Taurus

cluster. The regression effects arise from the circumstance that the coefficients in the observational equations involve the inaccurately known proper motions of the stars.

Formulae previously developed are extended to a more accurate solution of Charlier's equations for a cluster. Solutions by Merriman and Hertzsprung, for the two unknown coefficients in a linear relation between two variables x and y both subject to error, are shown to be consistent with a solution obtained in part I. *T. E. Sterne.*

Pignedoli, Antonio. Configurazioni ellissoidali di una massa continua disgregata e stratificata soggetta alla propria gravitazione e a quella di più centri lontani. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **79**, 332–345 (1944). [MF 16236]

The author studies the motion of a continuous medium subject to internal gravitational forces and external gravitational attraction by point masses. [Cf. Levi-Civita, Scritti Matematici Offerti a Luigi Berzolari, Pavia, 1936, pp. 161–168.] It is assumed that the motion takes place in planes perpendicular to a fixed axis, with force components parallel to the axis neglected. It is shown, in particular, that a rigid body motion about the axis is possible only for an ellipsoidal mass. *W. Kaplan* (Ann Arbor, Mich.).

Agostinelli, Cataldo. Nuovi contributi alla teoria degli anelli di Saturno. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **79**, 346–370 (1944). [MF 16237]

The rings of Saturn are studied on the basis of the Levi-Civita model of a continuum subject to internal gravitational forces. [Cf. the preceding review.] It is shown that a rotational motion is possible only when the mass is distributed in a plane through the center of mass O of Saturn. A "normal" motion is then studied, for which each particle moves with constant angular velocity on a circle of center O; the density is determined as a function of distance from O and the angular velocity. The stability of the motion is analyzed; upper limits for the density of a stable motion are determined, the limits being dependent on the frequency of the disturbance. *W. Kaplan* (Ann Arbor, Mich.).

Agostinelli, Cataldo. Sulla variazione dell'inclinazione del piano dell'orbita, del parametro e dell'eccentricità nel moto relativo di un pianeta puntiforme attratto colla legge di Armellini da un sole esteso e rotante. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **77**, 30–46 (1942). [MF 16248]

Continuing a previous investigation [Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **2** (1941)], the author studies the motion of a planet about the sun under the law of motion proposed by Armellini [Atti Accad. Naz. Lincei, Rend. Cl. Sci. Fis. Mat. Nat. (6) **26**, 209–215 (1937)], which gives rise to a damping term and a term dependent on the solar angular velocity. The variation of the eccentricity of the orbit and of the angle of inclination of the orbital plane with the plane perpendicular to the sun's axis of rotation are studied; the eccentricity is shown to decrease with increasing time, while the angle of inclination may increase for an interval, but will eventually decrease to zero. *W. Kaplan* (Ann Arbor, Mich.).

Agostinelli, Cataldo. Sulla variazione dell'ora terrestre. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **77**, 130–153 (1942). [MF 16250]

The author gives an explanation of a slight periodic irregularity in the earth's angular velocity about its axis in terms of a tidal effect. A simplified mechanical model of the earth, its oceans, the sun and the moon is used, and it is shown that the presence of the oceans gives rise to an angular acceleration which can account for the observed phenomenon. *W. Kaplan* (Ann Arbor, Mich.).

Dive, Pierre. Détermination du potentiel d'attraction à l'extérieur d'un astre par la pesanteur à sa surface. Arch. Sci. Phys. Nat., Geneva **23**, 169–172 (1941). [MF 14191]

Extending some earlier considerations of Stokes, the author shows that the gravitational potential exterior to an arbitrary rotating configuration is uniquely determined by the form of the bounding surface of zero pressure and the distribution of forces acting on this surface. *S. Chandrasekhar* (Williams Bay, Wis.).

Strömgren, Bengt. Tables of model stellar atmospheres (Model stellar atmospheres. I). Danske Vid. Selsk. Math.-Fys. Medd. **21**, no. 3, 85 pp. (1944). [MF 15404]

In this paper the structure of model stellar atmospheres characterized by various values of the parameters, the effective temperature T_e (determining the outward net flux of energy per unit area), the value of gravity g and the hydrogen metal ratio A are considered. The models studied cover the range of effective temperature T_e from that of G0 stars like the Sun ($T_e=5740°$) to that of A5 stars ($T_e=8500°$). The range of effective gravity covered corresponds to a region in the Hertzsprung-Russell diagram, roughly between a line somewhat below the main series and a line around $M_{bol}=0^m$, that is, slightly above the giant branch. The hydrogen-metal ratio was varied between the limits $\log A=3.4$ and $\log A=4.2$. In addition, a few special model atmospheres are also considered appropriate for the Sun, Capella and a cA5 star.

In the construction of the model stellar atmospheres it is assumed that the only sources of continuous opacity are hydrogen and the negative hydrogen ion. Furthermore, among the "metals" the elements magnesium, silicon, iron, calcium, aluminum and sodium are considered with the following relative abundances by numbers: 30:33:30:2:3:2, respectively. With these assumptions, the equation of hydrostatic equilibrium, together with the distribution of temperature (given by the theory of radiative transfer in the outer layers and by the adiabatic lapse-rate in the hydrogen convection zone) make the structure of the atmosphere determinate.

Various tables governing the march of total pressure, electron pressure and opacity are provided for the various model atmospheres considered. [It should perhaps be pointed out that the cross sections of the negative hydrogen ion used in the present paper have been superseded by more recent evaluations [S. Chandrasekhar, Astrophys. J. **102**, 395–401 (1945)].] *S. Chandrasekhar.*

Chandrasekhar, S. On the radiative equilibrium of a stellar atmosphere. X. Astrophys. J. **103**, 351–370 (1946). [MF 16783]

[For part IX of this series, cf. the same J. **103**, 165–192 (1946); these Rev. **7**, 489.] A further application of the author's theory developed particularly in parts II, III [same J. **100**, 76–86, 117–127 (1944); these Rev. **6**, 76, 190]. The radiative equilibrium of an atmosphere in which the Thomson scattering by free electrons governs the transfer of radiation is worked out, account being taken of the

polarization of the scattered radiation. The equations of transfer for the intensities referring to the two states of polarization in which the electric vector vibrates in the meridian plane and at right angles to it are separately formulated and solved to the nth approximation. Different laws of darkening are found for the two states of polarization so that the emergent radiation is polarized. The degree of polarization must vary from zero at the center of the disc to 11% at the limb. *G. C. McVittie* (London).

Bhatnagar, P. L. Anharmonic pulsations of a homogeneous star: effect of the ratio of specific heats. Bull. Calcutta Math. Soc. 38, 34–38 (1946). [MF 16971]

Roy, S. K. Rotational distortion of gaseous stars. Proc. Benares Math. Soc. (N.S.) 6, 49–66 (1944). [MF 16159]

The distortion of a polytrope of index 1 for uniform rotation is carried to the second order in the parameter $v=\omega^2/2\pi G\lambda$, where ω denotes the angular velocity, G the constant of gravitation and λ the central density. To this order the equation of the boundary of the configuration includes the fourth harmonic and has the form

$$\xi=\xi_1\{(1+v+9.61v^2)-(2.5v+13.85v^2)P_2(\mu)+2.49v^2P_4(\mu)+O(v^3)\},$$

where $\xi_1(=\pi)$ denotes the boundary of the undistorted polytrope and μ the direction cosine with respect to the axis of rotation. On the basis of this investigation the author concludes that the formation of a "furrow" by rotation may more easily happen for compressible configurations than for an incompressible fluid. *S. Chandrasekhar.*

Prasad, Chandrika. Configurations of a rotating incompressible mass of fluid surrounded by an atmosphere of small density. Proc. Benares Math. Soc. (N.S.) 6, 35–48 (1944). [MF 16158]

In this paper the rotational distortion of configurations consisting of an incompressible core of density $(1+s)\rho_0$ and an envelope of small but finite density $s\rho_0$ is considered. It is indicated that, in determining the shape of the boundary of the configuration, it is justifiable to neglect the distortion of the core, while it is necessary to consider the existence of the atmosphere in determining the shape of the core. With this simplification, the author is able to find an explicit solution for the problem considered. Numerical examples are given. *S. Chandrasekhar.*

Gasser, A. Entstehung, Aufbau, Energiehaushalt und Alter der Sterne und ihrer Planeten. Mitt. Naturwiss. Ges. Winterthur 24, 3–58 (1945). [MF 15900]

Gasser, A. Entstehung, Aufbau, Energiehaushalt und Alter der Sterne und ihrer Planeten. Helvetica Phys. Acta 18, 226–230 (1945).

In spite of the large claims made the first paper is unsound in principle [the second paper is a summary]. It is sufficient to point out, for example, that the extensive use which is made of the virial theorem in the form $2T+\Omega=0$ for every gaseous shell in an equilibrium configuration is wrong. When there is external pressure the correct form of the equation is

$$\int\left[\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2+r\cdot\text{grad }V\right]dm=-\int r\cdot\text{grad }PdV,$$

where V denotes the gravitational potential, dm the element of mass and dV the element of volume.
S. Chandrasekhar (Williams Bay, Wis.).

Hydrodynamics, Aerodynamics, Acoustics

Behrbohm, H., und Pinl, M. Zur Theorie der kompressiblen Potentialströmungen. I. Neue Linearisierung der Grundgleichung der ebenen adiabatisch kompressiblen Potentialströmung. Z. Angew. Math. Mech. 21, 193–203 (1941). [MF 15851]

The similarity between the quasi-linear differential equations of minimal surfaces and of plane adiabatic compressible potential flows is exploited to obtain a linearization of the latter by means of a generalized support function, which is analogous to that known for the former. The variational problem of the flow and its adjoint variational problem lead to two Euler's equations for the extremal surfaces z and $\bar{z}$, where $z=z(x, y)$ is the velocity potential of the flow. If $\omega(\alpha, \beta, \gamma)$ is a properly chosen generalized Minkowski support function for z, and $\bar{\omega}(\bar{\alpha}, \bar{\beta}, \bar{\gamma})$ that for $\bar{z}$, then the two equations have the following linearizations, respectively: (1) $\omega_{\alpha\alpha}+\omega_{\beta\beta}+C\omega_{\gamma\gamma}=0$, (2) $\bar{\omega}_{\bar{\alpha}\bar{\alpha}}+\bar{\omega}_{\bar{\beta}\bar{\beta}}+\bar{C}\bar{\omega}_{\bar{\gamma}\bar{\gamma}}=0$, C, $\bar{C}$ being known coefficients. The linearization (2) is achieved without knowledge of the Euler equation in its original form. The step from the solution of (1) to the flow plane is made by the relation grad $\omega=(x, y, z)$. *D. Gilbarg.*

Jacob, Caius. Sur une méthode d'approximation en mécanique des fluides compressibles. C. R. Acad. Sci. Paris 222, 1329–1331 (1946). [MF 16739]

The author gives a method of successive approximations for solving the problems of two-dimensional potential flows of an isentropic gas. Let $(V\cos\theta,\ V\sin\theta)$ be the components of velocity along the directions of the Cartesian axes (x, y) and let the velocity potential φ and the stream function ψ be defined by

$$(1)\qquad V\cos\theta=\frac{\partial\varphi}{\partial x}=\frac{\rho_0}{\rho}\frac{\partial\psi}{\partial y},\quad V\sin\theta=\frac{\partial\varphi}{\partial y}=-\frac{\rho_0}{\rho}\frac{\partial\psi}{\partial y},$$

where ρ is the density and ρ_0 is its value at the stagnation point. Then, by choosing $f=\varphi+i\psi$ and $\bar{f}=\varphi-i\psi$ as independent variables, and $\omega=\theta+i\log V$ and $\bar{\omega}=\theta-i\log V$ as dependent variables, one obtains

$$(2)\qquad \left(\frac{\rho_0}{\rho}-1\right)\frac{\partial\omega}{\partial f}+\left(\frac{\rho_0}{\rho}+1\right)\frac{\partial\omega}{\partial\bar{f}}=\frac{1}{2}\frac{\rho_0}{\rho}\frac{V^2}{c^2}\left(\frac{\partial\omega}{\partial f}+\frac{\partial\omega}{\partial\bar{f}}-\frac{\partial\bar{\omega}}{\partial f}-\frac{\partial\bar{\omega}}{\partial\bar{f}}\right),$$

where c is the velocity of sound. To transform back to the plane $z=x+iy$, one has the relation

$$(3)\qquad dz=\tfrac{1}{2}e^{i\omega}(1+\rho_0/\rho)df+\tfrac{1}{2}e^{i\omega}(1-\rho_0/\rho)d\bar{f}.$$

The author then proposes to solve the equation (2) by expressing ρ_0/ρ, ω and $\bar{\omega}$ in powers of $M^2=U^2/c_0^2$, where U is the velocity at infinity and c_0 is the velocity of sound in the fluid at rest. Detailed results are to be presented elsewhere. *C. C. Lin* (Providence, R. I.).

Chaplygin, S. Gas jets. Tech. Memos. Nat. Adv. Comm. Aeronaut., no. **1063**, 112 pp. (3 plates) (1944). [MF 16324]

Translation of a paper which appeared in Učenye Zapiski Imp. Moskov. Univ., Otd. Fiz.-Mat. 21, 1–121 (1904).

Theodorsen, Theodore. Extension of the Chaplygin proofs on the existence of compressible-flow solutions to the supersonic region. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1028**, 8 pp. (1 plate) (1946). [MF 15756]

Results obtained by S. Chaplygin in the paper quoted above are extended from the subsonic region to the adjacent supersonic region, provided the Jacobian of the transformation from the state plane to the (φ, ψ)-plane does not vanish in the region in question. *M. H. Martin.*

Frankl, F. On the problems of Chaplygin for mixed sub- and supersonic flows. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **9**, 121–143 (1945). (Russian. English summary) [MF 12769]

The author considers steady two-dimensional irrotational flows of a compressible fluid in the (H, θ) plane, where $\mathrm{H}=\mathrm{H}(q)$ is given by $d\mathrm{H}/dq=\rho/q$, ρ being the density and q the speed. The equation for the stream function assumes in this plane the form $l(\mathrm{H})\psi_{\theta\theta}+\psi_{\mathrm{HH}}=0$. [See, for example, Bergman, The Hodograph Method in the Theory of Compressible Fluids, Brown University, 1942, formulas (3.8) and (8.18).] Let $\mathrm{H}_0=\mathrm{H}(q_0)$, where q_0 is the velocity of sound. For $\mathrm{H}<\mathrm{H}_0$ the motion is subsonic; for $\mathrm{H}>\mathrm{H}_0$ it is supersonic. Let S denote a domain which includes a segment CA of the θ-axis in its interior and consists of two parts, S_1 situated in the subsonic region and bounded by the curve $ACBA$ (AC being a segment of the line $\mathrm{H}=\mathrm{H}_0$), and S_2 situated in the supersonic region and bounded by AC and two characteristics CD and DA. The author proves the following uniqueness theorem. If a solution of the equation $l\psi_{\theta\theta}+\psi_{\mathrm{HH}}=0$ vanishes on the curve CBA and on either one of the two characteristics, say CD, then it is identically zero in $S=S_1+S_2$, provided that the arc AC of the line $\mathrm{H}=\mathrm{H}_0$ is not larger than $\theta_0\doteq 3\pi/10$. Applications of this result to the investigation of flows in nozzles are discussed. Two theorems proved by the author should be mentioned. (1) Let $s_\nu(\tau)$ denote functions introduced by Chaplygin [formula (14) of his paper quoted above]. Generalizing a result by Chaplygin, the author proves that for $\tau=1/(2\beta+1)$, that is, on the line of the transition from the subsonic to the supersonic region, the asymptotic formula

$$[d \log s_\nu(\tau)/d\tau]_{\tau=(2\beta+1)^{-1}}=C_\nu^{\frac{1}{3}}+O(1)$$

holds. (2) Let ψ be a bounded solution of the equation $\psi_{\mathrm{HH}}+l(\mathrm{H})\psi_{\theta\theta}=0$ which is defined in S_1, $\overline{AC}=\theta_0$. Let, furthermore, $\psi(\mathrm{H}, 0)=\psi(\mathrm{H}, \theta_0)=0$. Then there exists a kernel $K(\theta, \theta')$ which is independent of ψ, and its singular part is given by

$$K(\theta, \theta')=A\{|\theta-\theta'|^{-\frac{1}{3}}+(\theta+\theta')^{-\frac{1}{3}}+(\theta-\theta'-2\theta_0)^{-\frac{1}{3}}\}+O(1),$$

which makes it possible to express the boundary values of ψ on $\mathrm{H}=\mathrm{H}_0$ in terms of the boundary values of $\psi_{\mathrm{H}}(=\partial\psi/\partial\mathrm{H})$, for $\mathrm{H}=\mathrm{H}_0$, namely,

$$\psi(\mathrm{H}_0, \theta)=\int^{\theta_0} K(\theta, \theta')\psi_{\mathrm{H}}(\mathrm{H}_0, \theta')d\theta'.$$

S. Bergman (Cambridge, Mass.).

Brown, John P. The stability of compressible flows and transition through the speed of sound. AAF Technical Report no. **5410**, 73 pp. (1946). [MF 16320]

Theoretical calculations on isentropic compressible flows without viscosity indicate the existence of mixed subsonic and supersonic flows, involving not only a smooth transition from subsonic to supersonic velocities but also a smooth transition from supersonic to subsonic velocities. Experimental ob ervations, on the other hand, generally show that the transitions from supersonic velocities to subsonic velocities are not smooth but are accompanied by shock waves. Two explanations for this discrepancy are advanced: that the breakdown of the isentropic flow is due to the interaction of boundary layer and the shock wave, that is, the viscosity effect; and that the inviscid flow itself is unstable. The author investigates the latter possibility. The particular problem which is treated by the author in detail is the stability of flow in a straight channel of slowly varying cross section. This problem can be analyzed in two independent variables, the time and the distance along the axis of the channel. In this approximation, the surface of constant velocity is perpendicular to the flow direction. By linearizing the differential equations, the author proves that small disturbances generated in the decelerating region of supersonic flow are propagated downstream towards the surface of Mach number 1 with increasing amplitude. Similarly, disturbances generated in the decelerating region of subsonic flow are propagated upstream towards the surface of Mach number 1 with increasing amplitude. Therefore, the transition from supersonic flow to subsonic flow in such a channel is definitely unstable. For accelerating flows, the reverse is true and the transition from subsonic flow to supersonic flow is thus stable.

The author then surmises that in general if for any steady solution of two-dimensional or three-dimensional inviscid compressible flow the surface of Mach number 1 is perpendicular to the stream direction in the decelerating region, the flow is unstable and shock waves will appear. Since this situation will occur generally only after a rather extensive supersonic region has developed in the transsonic flows, the conclusion by the author is that the isentropic flow over a solid body will not break down with the first appearance of local Mach number unity in the flow. *H. S. Tsien.*

Pretsch, J. Über die Stabilität einer Laminarströmung in einem geraden Rohr mit kreisförmigem Querschnitt. Z. Angew. Math. Mech. **21**, 204–217 (1941). [MF 15852]

The author studies the stability problem of axially symmetrical laminar flows under axially symmetrical disturbances. The mathematical method used is similar to that used for the study of two-dimensional parallel motions. From these investigations, the author concludes that the laminar flow through a circular pipe (Hagen-Poiseuille flow) is stable. The discussion is in line with that of T. Sexl [Ann. Physik (4) **83**, 835–848; **84**, 807–822 (1927)].

C. C. Lin (Providence, R. I.).

Görtler, H. Instabilität laminarer Grenzschichten an konkaven Wänden gegenüber gewissen dreidimensionalen Störungen. Z. Angew. Math. Mech. **21**, 250–252 (1941). [MF 15857]

Short résumé of a paper published before [Nachr. Ges. Wiss. Göttingen. Fachgruppe I. (N.F.) **2**, 1–26 (1940); these Rev. **2**, 267]. *H. W. Liepmann* (Pasadena, Calif.).

Urano, Kaoru, and Munakata, Ken-iti. On the stability of a double row of vortices with arbitrary stagger. Proc. Phys.-Math. Soc. Japan (3) **24**, 790–799 (1942). [MF 15043]

The authors re-examine the stability of a double row of vortices in a nonviscous fluid, in view of a discrepancy in the results of Rosenhead [Proc. Cambridge Philos. Soc. **25**, 132–138 (1929)] and Maue [Z. Angew. Math. Mech. **20**, 129–137 (1940); these Rev. **2**, 170]. Rosenhead concluded

that the Kármán street is the only stable configuration; Maue found that the Kármán street is not the only stable configuration.

Consider vortices of strength $k=2\pi$ placed at the points $(m+\frac{1}{2}d, \frac{1}{2}h)$ and $(n-\frac{1}{2}d, -\frac{1}{2}h)$, respectively. The velocity potential of this configuration is

$$\phi=-i\log\sin\{\pi(z-z_0)\}/\sin\{\pi(z+z_0)\}$$

with $z_0=\frac{1}{2}d+\frac{1}{2}h$. Consider now the vortices displaced to $(n-\frac{1}{2}d+x_n', -\frac{1}{2}h+y_n')$ and $(m+\frac{1}{2}d+x_m, \frac{1}{2}h+y_m)$. The authors first obtain the equations for dx_m/dt, dy_m/dt, etc. from the velocity potential. The displacements are then chosen of the form $x_r=\alpha\cos(r+d)\varphi+\gamma\sin(r+d)\varphi$ and similarly for y_r, x_s', y_s'; α, γ are real functions of the time. Inserting these expressions into the equations for dx_m/dt, etc., furnishes a set of linear differential equations for the $\alpha(t)$, $\gamma(t)$, etc. The motion is found to be stable if $\sin\pi d=\sinh\pi h$. The results are found to agree with Maue. The authors conclude that Rosenhead's results are in error.

H. W. Liepmann (Pasadena, Calif.).

Tsien, Hsue-shen. The "limiting line" in mixed subsonic and supersonic flow of compressible fluids. Tech. Notes Nat. Adv. Comm. Aeronaut., no. 961, 29 pp. (4 plates) (1944). [MF 14399]

The chief results of Tollmien, Ringleb and others on the theory of the limiting lines in two-dimensional supersonic flows are extended here to the case of three-dimensional flows. It is shown for axially symmetric flows that (1) the limiting line is an envelope of Mach lines in the (meridian) flow plane; (2) the flow undergoes infinite accelerations and reverses direction at the limiting line; (3) in the hodograph plane, the streamlines and one family of characteristics are tangent to each other at the limiting line; (4) the flow cannot be extended beyond the limiting line without a breakdown of isentropic irrotational flow. Corresponding proofs are sketched for general three-dimensional flows.

D. Gilbarg (Bloomington, Ind.).

Feinsilber, A. M. Fundamental problems of fluid dynamics for boundary layer on airplane wing. C. R. (Doklady) Acad. Sci. URSS (N.S.) **48**, 86–88 (1945). [MF 15222]

The author considers the problem of boundary layers, including the case of compressible and turbulent flow. [The method of approach is based on two papers by Leibenson and Golubev which were unavailable to the reviewer.] Explicit results for the turbulent boundary layer of compressible fluid flow past a flat plate are given.

H. W. Liepmann (Pasadena, Calif.).

Kochin, N. E. Theory of the circular wing. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **9**, 13–66 (1945). (Russian. English summary) [MF 13504]

This memoir treats the flow of an incompressible fluid past an infinitely thin slightly curved airfoil given by the equation $z=\zeta(x,y)$, $x^2+y^2\leq a^2$. This problem was treated previously by Kinner [Ing.-Arch. **8**, 47–80 (1937)] and by the author [J. Appl. Math. Mech. **4**, 3–32 (1940); Appl. Math. Mech. **6**, 287–316 (1942); these Rev. **4**, 228]. In the present paper a complete, rigorous and effective solution is given for the steady case, $\zeta(x,y)$ being an arbitrarily prescribed function. The usual "linearization" of the boundary conditions leads to the following problem: to determine a harmonic function $\varphi(x,y,z)$ defined for all values of x, y, z except those given by (S) $z=0$, $x^2+y^2\leq a^2$ and (Σ) $z=0$, $x^2+y^2>a^2$, $|y|\leq a$, $x<0$, and such that (i) grad φ is bounded at the trailing edge ($x^2+y^2=a^2$, $x<0$) and becomes infinite like $\delta^{\frac{1}{2}}$ at the leading edge ($x^2+y^2=a^2$, $x>0$), δ being the distance from the leading edge, (ii) grad φ vanishes at infinity, (iii) on the disk S: $\partial\varphi/\partial z=-c\partial\zeta/\partial x$, (iv) on the "vortex sheet" Σ: $\partial\varphi/\partial x=0$, (v) on the remaining part of the plane $z=0$: $\varphi=0$.

The author obtains the solution in the form

$$\varphi=\varphi_0+\sum C_nH_n;$$

φ_0 is the solution of the classical Neumann problem $(\varphi_z=-c\zeta_z)$ for the domain exterior to the disk S; H_0 is the potential of a purely circulatory flow around S due to a vorticity distribution on S and such that Σ is free of vorticity except for the two vortex lines $y=\pm a$; H_n $(n=1,2,\cdots)$ is the potential of a purely circulatory flow past S such that the vorticity distribution across Σ is given by the formula $\Gamma=P_n(y/a)$, P_n being the nth Legendre polynomial; φ_0 is given by the formula

$$\varphi_0=\tfrac{1}{2}\pi^{-1}\int\int K(x,y,z,\xi,\eta)f(\xi,\eta)d\xi d\eta,$$

where $f=c\partial\varphi/\partial x$ and K is an explicitly given kernel. The functions H_n are given by rather complicated formulas involving associated Legendre functions of both kinds. The coefficients C_n must be determined so that φ shall satisfy all boundary conditions, notably the Joukowsky condition at the trailing edge. This leads to a system of infinitely many linear equations, with coefficients depending upon the shape of the wing, that is, upon the function ζ. An explicit solution of this system is obtained in the form of rapidly converging infinite series. The lift, induced drag and the moments of the pressure forces can be expressed in terms of the coefficients C_n, the lift depending only on C_0.

The author proves the validity of his solution and indicates that the amount of computational labor involved is reasonable. Three numerical examples are given.

L. Bers (Pittsburgh, Pa.).

Bers, Lipman. On a method of constructing two-dimensional subsonic compressible flows around closed profiles. Tech. Notes Nat. Adv. Comm. Aeronaut., no. 969, 61 pp. (4 plates) (1945). [MF 16302]

By using the tangent to the isentropic pressure-volume curve as an approximation to the isentropic curve itself, it was found by von Kármán and Tsien [J. Aeronaut. Sci. **6**, 399–407 (1939)] that an incompressible flow around a profile can be transformed into a compressible flow around a similar profile. Their method is restricted, however, to flows without circulation. The author makes a generalization of the von Kármán-Tsien method to flows with circulation. If q is the speed of compressible flow, then the author defines a distorted speed q^*, similar to that of previous investigators, as $q^*=q/[1+(1+q^2)^{\frac{1}{2}}]$. If θ is the angle of inclination of the velocity vector with respect to the x-axis, the components u^* and v^* of the distorted speed are given by $u^*-iv^*=q^*e^{-i\theta}$. The complex potential for the compressible flow is then an analytic function of the variable u^*-iv^*. Now, if the conformal mapping of the compressible flow in the u^*-v^* plane to the incompressible flow in the hodograph plane of u_i and v_i is given by $u_i-iv_i=(u^*-iv^*)^n$ and if $n=(1-M_\infty^2)^{-\frac{1}{2}}$, M_∞ the free-stream Mach number, the closed contour for the incompressible flow will map into a closed contour for the compressible flow, even with circulation. The relations for calculating the pressure and velocity can then be easily deduced. For ordinary profiles with

rounded nose and a sharp trailing edge, the condition of simple connectedness in the $u_i - v_i$ plane requires $n<2$ or $M_\infty < \sqrt{\frac{2}{3}}$.

Some comments are presented by the author for the more general case of using the actual isentropic relation.

H. S. Tsien (Cambridge, Mass.).

Bers, Lipman. On the circulatory subsonic flow of a compressible fluid past a circular cylinder. Tech. Notes Nat. Adv. Comm. Aeronaut., no. 970, 30 pp. (12 plates) (1945). [MF 14402]

The author's method for constructing two-dimensional subsonic compressible flows around closed profiles with the assumption of ratio of specific heats $\gamma=-1$ [see the preceding review] is applied to the circulatory flows past a circular cylinder. Due to the thickness of the body, the subsonic flows are limited to rather small free stream Mach numbers. In this case, the von Kármán-Tsien velocity correction formula, which is theoretically applicable only for flows without circulation, is found to yield results closely approximating the present calculations but generally slightly higher.

The author interprets the assumed pressure-density relation as a tangent to the exact isentropic relation at a point corresponding to the stagnation condition; von Kármán and Tsien interpreted it as a tangent at a point corresponding to the free stream condition. The author rightly states that their relative merits can be determined only by comparison with rigorous solutions. The close check of the von Kármán-Tsien formula with both rigorous theoretical solutions and with experimental data seems to justify their interpretation of the assumed pressure-density relation. *H. S. Tsien.*

Manwell, A. R. Expansion in series of the exact solution for compressible flow past a circular or an elliptic cylinder. Philos. Mag. (7) 36, 499–510 (1945). [MF 15952]

The desired solutions are expressed as expansions in (essentially) the Chaplygin hodograph solutions, the unknown coefficients in the expansions being outlined by fitting the boundary conditions. This is done by introducing two auxiliary power series in the stream velocity which are related to each other and to the solution through the boundary conditions. Relations among the coefficients of these series are established, and from these it is possible to obtain the velocity distribution along the boundary. The details are carried out for isothermal and adiabatic flows past circular and elliptic cylinders. Convergence is not rapid, and, as a numerical example shows, terms up to the seventh power in the stream velocity are required for an accurate value of the critical Mach number. The procedure in satisfying the boundary conditions is quite laborious, so that the method as a whole is no simplification over previous expansion methods. *D. Gilbarg* (Bloomington, Ind.).

Risack, M. Note sur la nature du champ des vitesses autour d'un profil en fluide parfait et en fluide naturel. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 29, 93–107 (1943). [MF 13840]

The author first considers the field of the Laplacian of the velocity and then goes on to discuss the formation of vorticity in a perfect fluid and a viscous fluid. Finally, he considers the field of velocity around a profile both in the case of a perfect fluid and in the case of a viscous fluid. In the former case, the vorticity and Laplacian vanish everywhere. In the latter case, vorticity and Laplacian exist everywhere. *C. C. Lin* (Providence, R. I.).

Brun, Edmond, et Vasseur, Marcel. Écoulements laminaires dans le cas où la viscosité du fluide varie suivant le lieu. C. R. Acad. Sci. Paris 219, 573–575 (1944). [MF 15290]

The authors discuss laminar flow in which the viscosity varies because of changes in temperature. The laminar stress terms in the Navier-Stokes equations are thus of the form

$$\frac{\partial \tau_{ik}}{\partial x_k}=\frac{\partial}{\partial x_k}\left(\mu\frac{\partial u_i}{\partial x_k}\right).$$

The case of pressure flow between parallel plates of different temperatures is outlined. *H. W. Liepmann.*

van Wijngaarden, A. Flow in the radial direction between two plane surfaces. Nederl. Akad. Wetensch. Verslagen, Afd. Natuurkunde 52, 29–36 (1943). (Dutch. German, English and French summaries) [MF 15336]

The author investigates laminar flow of an incompressible viscous liquid in the space bounded by two parallel planes and a circular cylinder perpendicular to them. Methods familiar in boundary layer theory are adapted to this problem. The partial differential equation of the stream function is reduced to an infinite system of ordinary differential equations. An asymptotic solution of this system is given which is valid in the vicinity of the cylindrical boundary.

P. Neményi (Pullman, Wash.).

Robin, Louis. Étude de l'énergie cinétique d'un liquide visqueux incompressible emplissant l'espace, quand le temps croit indéfiniment. J. Math. Pures Appl. (9) 24, 33–49 (1945). [MF 15967]

The author discusses the asymptotic behavior of the kinetic energy of a nonstationary viscous flow for the case when the time t approaches infinity. The linearized equations are considered; in the first part the nonlinear terms in the Stokes-Navier equation are neglected (Stokes approximation); in the second part the nonlinear terms are carried in first approximation (Oseen approximation). The equations are thus of the type of the heat-conduction equation and solutions for the velocity components $u_i(x_k, t)$ are of the form

$$u_i(x, t)=\tfrac{1}{8}(\pi\nu t)^{-\frac{3}{2}}\int\!\!\int\!\!\int e^{-x^2/4\nu t}u_i(y, 0)dy$$

for the case of Stokes's approximation and similar but more complex for Oseen's approximation. The author assumes the inequality

$$\{u_j(y, 0)u_j(y, 0)\}^{\frac{1}{2}} < U/(1+\beta R)^{\frac{3}{2}+\alpha}$$

to hold, where U is the potential of the external forces, $R=(y_j y_j)^{\frac{1}{2}}$ and α and β are positive constants. The behavior of the kinetic energy as $t\to\infty$ is then discussed for various values of α. For example, for three-dimensional motion satisfying the Stokes approximation the author finds, for the kinetic energy $I(t)$,

$$\begin{aligned} &I(t) < A\,U\beta^{-\frac{3}{2}-\alpha}(\nu t)^{-\alpha/2}, && \alpha<\tfrac{3}{2},\\ &I(t) < A\,U\beta^{-3}(\nu t)^{-\frac{3}{4}}, && \alpha>\tfrac{3}{2},\\ &I(t) < A\,U\log(\nu t)\beta^{-3}(\nu t)^{-\frac{3}{4}}, && \alpha=\tfrac{3}{2}, \end{aligned}$$

where A is a dimensionless constant.

Similar results are obtained for two-dimensional motion and also for the case where the Oseen approximation is used.

H. W. Liepmann (Pasadena, Calif.).

Tandon, H. S. On a property of λ. Proc. Benares Math. Soc. (N.S.) 6, 1–2 (1944). [MF 16155]

The author shows that, if the motion of a viscous homogeneous incompressible fluid has the properties of being steady, of satisfying Bernoulli's equation, and of having the stream lines and vortex lines coincident, then the motion is irrotational. *C. C. Torrance* (Annapolis, Md.).

Cârstoiu, I. Théorie des tourbillons dans un fluide visqueux. Bull. École Polytech. Bucarest [Bul. Politehn. București] 14, 27–50 (1943). [MF 13574]

The vortex theorems in a viscous fluid are studied in analogy to those in a perfect fluid. In general, the results are complicated by the viscous terms in the Navier-Stokes equations. The author stresses some known results such as (i) a potential flow remains potential in a viscous fluid under conditions similar to those required in a perfect fluid (provided that there is no solid boundary) and (ii) the Biot-Savart formula holds for a viscous incompressible fluid (the formula being purely kinematic). Using this last formula, he writes down the integro-differential equations of motion and proposes to discuss later the existence theorem in a viscous fluid along the lines of Lichtenstein for perfect fluids. *C. C. Lin* (Providence, R. I.).

Carrier, G. F., and Carlson, F. D. On the propagation of small disturbances in a moving compressible fluid. Quart. Appl. Math. 4, 1–12 (1946). [MF 15942]

A general stream is considered. The propagation of small disturbances in it is analyzed by applying standard perturbation techniques to the Navier-Stokes and continuity equations and then treating the derived equations by the theory of characteristics. A simplification is achieved by replacing the energy equation by the condition that the changes in state from undisturbed to disturbed stream are isentropic; this assumption is justified by an order of magnitude argument. However, no restrictions are placed on the undisturbed stream, so that the analysis is otherwise quite general. It is shown that rotational disturbances move with the stream (except for diffusion), whereas the irrotational portions of a disturbance propagate relative to this general stream as they would in an irrotational flow. Specific application is made to the propagation of pulses through a simplified boundary layer into a uniform stream. The wave fronts thus predicted are found to be in qualitatively good agreement with those observed in Schlieren photographs. *D. Gilbarg* (Bloomington, Ind.).

Omara, M. A. Extension of the theory of single burbling to general plane motion. Proc. Math. Phys. Soc. Egypt 2, no. 2, 41–54 (1944). [MF 16339]

In this paper the theory of single burbling, developed by Witoszynski in 1922 for a uniform stream past a fixed profile, has been extended to the case of the general uniplanar motion of a profile in an infinite mass of fluid of uniform density. The resultant force and moment are calculated, by means of the generalized Blasius contour integrals and by using appropriate contours, in terms of the coefficients of the expansion of the function transforming conformally the exterior of the profile into the exterior of a circle of unit radius. The consideration of the cusped profiles leads to the addition of certain terms to the expressions of the resultant force and moment. These terms are interpreted as representing forces acting each along the tangent to the profile at the corresponding cusp. Application to the case of a flat plate leads to a generalization of a result obtained by Witoszynski, namely: the resultant pressure is always normal to the plane. *Author's summary.*

Loitsianskii, L. G. Some basic laws of isotropic turbulent flow. Tech. Memos. Nat. Adv. Comm. Aeronaut., no. 1079, 36 pp. (1945). [MF 13442]

[Translation of Report no. 440 of the Central Aero-Hydrodynamical Institute, Moscow, 1939.] The author first reviews the theory of isotropic turbulence, as developed by von Kármán and Howarth [Proc. Roy. Soc. London. Ser. A. 164, 192–215 (1938)], but now presented in a somewhat different manner. He then derives the conservation of the disturbance moment and points out the analogy of the von Kármán-Howarth equation of propagation of correlations with the equation of heat conduction in the five-dimensional space (with terms of the triple correlation von Kármán-Howarth equation corresponding to convection terms). The conservation of disturbance moment then corresponds to the constancy of the total amount of heat.

Neglecting the terms involving triple correlations, the author solves the equation of propagation in a general manner, this simplified equation being exactly the equation of heat conduction in the five-dimensional space. *C. C. Lin* (Providence, R. I.).

Schmidt, Wilhelm. Turbulente Ausbreitung eines Stromes erhitzter Luft. I. Z. Angew. Math. Mech. 21, 265–278 (1941). [MF 15862]

The author considers the turbulent mixing of plane and axially symmetric hot-gas convection jets. The momentum transport theory of L. Prandtl is applied to both cases. In addition, Taylor's vorticity transport theory is applied to the plane jet. The present paper differs from most considerations of heated jets because here the buoyancy is of main importance; the air is set in motion by addition of heat only. The standard assumptions of similarity downstream, constant mixing length across the jet, etc., are made. With these assumptions the equations of motion can be integrated approximately. The difference between Prandtl's and Taylor's assumptions appears, as always, in the relative behavior of temperature and velocity profile. *H. W. Liepmann.*

Possio, Camillo. Sulla teoria del moto stazionario di un fluido pesante con superficie libera. Ann. Mat. Pura Appl. (4) 20, 313–329 (1941). [MF 16608]

Per rendere determinato il campo di moto stazionario di un fluido pesante con superficie libera, il'Rayleigh, com'è noto, introduce una fittizia forza d'attrito che fa poi tendere a zero: l'autore dimostra come si possa giungere allo stesso risultato, in modo più convincente, immaginando il moto stazionario come caso limite di un moto vario iniziatosi dalla quiete. *Author's summary.*

Batchelor, G. K. On the concept and properties of the idealized hydrodynamic resistance. Commonwealth of Australia. Council Sci. Ind. Res. Division Aeronaut. Rep. no. ACA 13, 15 pp. (1945). [MF 15543]

Essentially the same as Commonwealth of Australia. Council Sci. Ind. Res. Division Aeronaut. Rep. no. 955 (1944); these Rev. 7, 95. *C. C. Lin* (Providence, R. I.).

Leibenson, L. S. Les nouvelles équations de mouvement d'un liquide saturé par un gaz dans un milieu poreux. C. R. (Doklady) Acad. Sci. URSS (N.S.) 49, 172–174 (1945). [MF 16407]

Starr, Victor P. A quasi-Lagrangian system of hydrodynamical equations. J. Meteorol. 2, 227–237 (1945). [MF 16472]

For numerous hydrodynamical studies of stratified media it is convenient to use a coordinate system consisting of two space coordinates x and y and a material coordinate, the latter corresponding to the Lagrangian coordinates, while the two former correspond to the Eulerian coordinates. Since, in the atmosphere and the ocean, surfaces of equal property are nearly horizontal because of the effect of gravity, the material coordinate c replaces as a rule the vertical coordinate of the Eulerian system, although this choice is not essential. The quantity c should be a conservative property so that the equations do not become too complicated. The equation of continuity can be written in a number of different forms which are discussed at some length. In analogy to the stream function of the Eulerian system a stream function for material surfaces can be introduced. The equations of motion in such a system become particularly simple if the c surfaces can be chosen so as to make the density a function of the pressure alone along each surface of constant c. Finally it is shown how the vorticity theorem and the circulation theorem can be written in the coordinate system. *B. Haurwitz.*

Starr, Victor P. Note on individual pressure changes in surface waves. J. Meteorol. 3, 23–24 (1946). [MF 16593]

Rossby, C.-G. On the propagation of frequencies and energy in certain types of oceanic and atmospheric waves. J. Meteorol. 2, 187–204 (1945). [MF 16471]

The author studies certain types of plane waves whose frequency and wave number are slowly varying functions of space and time. It is assumed that the number of wave crests is conserved in which case a simple kinematic relation exists between the wave number and the frequency. If the physical characteristics of the wave motion are such that the phase velocity and wave length uniquely determine each other the waves possess a characteristic group velocity with which the frequencies are propagated. In the more general case, however, when the frequency equation contains space and time explicitly an observer travelling with the customarily defined group velocity will observe frequency changes, although these are small except in the leading and trailing boundaries of the wave trains, at least in the wave types which are considered. The three types specifically investigated are long waves on a rotating disc, surface waves of small amplitude in deep water and nondivergent plane waves in the westerlies of the atmosphere. For these the propagation of energy is studied. Considering planes moving with the ordinary group velocity it is found that the energy flux across these planes is nondivergent where frequency changes along these moving planes do not occur. The boundary regions where the frequency varies with time serve as sources and sinks of energy. Hence the classical energy-propagation equation indicates not that the energy is propagated with the speed of the group velocity, but rather that the energy transport relative to the group-velocity planes is nondivergent. The equations determining group-velocity propagation and nondivergent propagation of energy together form a closed system independent of the special dynamics of the wave motion under consideration and permit one to compute the future distribution of these elements from the initial conditions. The analysis of the waves in the westerlies gives an explanation of the fast-moving isallobaric waves which seem to emanate from the centers of action, inasmuch as the theory gives "forerunner" wave trains. Finally, in view of the results of the study the suggestion is made that the retrograde "blocking" observed in the westerlies might be interpreted as the effect of a convergent distribution of the group velocity in the long quasistationary waves in the westerlies. *B. Haurwitz.*

Burgers, J. M. On the one-dimensional propagation of pressure-disturbances in an ideal gas. Nederl. Akad. Wetensch. Verslagen, Afd. Natuurkunde 52, 476–484, 560–570 (1943). (Dutch. German, English and French summaries) [MF 15335]

A semi-infinite cylindrical tube is closed at one end by a movable piston, which is suddenly set in motion, moved at a constant velocity and then suddenly stopped. The author discusses the resulting motion of a nonviscous gas contained in the tube. The effects of heat conduction and radiation are neglected. The compression waves, expansion waves and shock waves generated are studied and a geometrical method is developed which makes it possible to give an approximate picture of the most essential features of the wave system developing through their interaction. *P. Neményi.*

***Rayleigh, John William Strutt, Baron. The Theory of Sound.** 2d ed. Dover Publications, New York, N. Y., 1945. Two volumes in one. Vol. I, xlii+480 pp.; vol. II, xii+504 pp. $4.95.

Photographic reproduction of the 1929 reprint of the edition of 1894–1896, published by Macmillan, London. A 28 page historical introduction by R. B. Lindsay has been added.

Miles, John W. The analysis of plane discontinuities in cylindrical tubes. I. J. Acoust. Soc. Amer. 17, 259–271 (1946). [MF 14672]

Miles, John W. The analysis of plane discontinuities in cylindrical tubes. II. J. Acoust. Soc. Amer. 17, 272–284 (1946). [MF 14673]

A horizontal cylindrical tube has a window at $x=0$ and (possibly) different uniform cross sections to the right or left. A plane sound wave is the incident disturbance. The pressure is $\phi(y, z)\exp(\pm ihx)$ and the well-known key equations are (1) $\nabla^2\phi+((2\pi/\lambda)^2-h^2)\phi=0$ and (2) $\partial\phi/\partial n=0$ at the cylinder boundary. The problem is essentially that of matching the solutions in both halves of the tube to satisfy the conditions at $x=0$ on the pressure and velocity. The author gives a transmission line paraphrasing which indicates the central importance of a certain lumped capacity. This is the minimum of an integral involving the unknown function quadratically (if the author's functions are suitably normalized). The Ritz method is applied to determine the coefficients in the expansion of the minimizing function in terms of the characteristic functions associated with (1) and (2). The variational integral is ascribed to Schwinger. Other problems of this sort are mentioned; the papers are in large part devoted to physical discussion.

D. G. Bourgin (Urbana, Ill.).

Patnaik, Brajabihari. Vibration of air-columns in closed pipes. Bull. Calcutta Math. Soc. 37, 137–140 (1945). [MF 16163]

This paper is closely related to an earlier one by Mohanty and Patnaik [same Bull. 36, 79–82 (1944); these Rev. 6, 74]. The author now discusses the case in which the pipe is closed at one or both ends. *L. A. MacColl.*

Brekhovskikh, L. M. Propagation of sound waves in a liquid layer between two absorbing half-spaces. C. R. (Doklady) Acad. Sci. URSS (N.S.) 48, 397–400 (1945). [MF 16653]

Elasticity, Plasticity

Sternberg, E. Nonlinear theory of elasticity with small deformations. J. Appl. Mech. 13, A-53–A-60 (1946). [MF 15737]

The paper is concerned with the problems of elasticity when the material is one whose stress-strain relation is nonlinear. The author assumes a linear approximation to the strain-displacement relations and arrives at the strain energy as a power series in the invariants of the strain tensor. Since he then retains powers of these functions which lead to quadratic terms in the stress-strain relations (the stresses being given by derivatives of the strain energy with respect to the strain components) the assumption that the energy is a function of invariants arising from the original linear approximation is not wholly consistent. However, for a suitable range of (small) displacement magnitudes, the results are probably quite indicative.

The equations of equilibrium, when combined with the stress-strain relations, lead to a set of nonlinear equations for the displacements. The boundary value problems will require approximation methods in general and first approximations are conveniently taken as the corresponding linear results. In the paper, results are obtained for torsion, uniform tension and uniform compression of circular cylinders. The results are discussed in detail. *G. F. Carrier.*

Oka, Syôten, und Ôkawa, Akiya. Über den Zusammenhang zwischen Spannung und Deformation isotroper Substanzen beim gemeinsamen Auftreten von Elastizität, Plastizität und Viskosität. Proc. Phys.-Math. Soc. Japan (3) 25, 406–412 (1943). [MF 15061]

The author discusses the mechanical behavior of isotropic solids which are characterized by stress-strain relations of the form $M(\sigma_{ij}-\frac{1}{3}\sigma_{kk}\delta_{ij})=N(\epsilon_{ij}-\frac{1}{3}\epsilon_{kk}\delta_{ij})$, $P\sigma_{kk}=Q\epsilon_{kk}$, where σ_{ij} and ϵ_{ij} are the tensors of stress and strain and M, N, P, Q are polynomials in the differential operator $\partial/\partial t$. *W. Prager* (Providence, R. I.).

Prager, W. Strain hardening under combined stresses. J. Appl. Phys. 16, 837–840 (1945). [MF 14542]

The purpose of this paper is to find a general stress-strain relation which in the case of simple tension or compression reduces to

$$e_1=c_1s_1+c_3s_1^3+c_5s_1^5+\cdots.$$

The quantities e_1, e_2, e_3 are the components of principal strain and the quantities s_1, s_2, s_3 are the components of principal stress deviation. Assuming incompressibility, that is, the relation $e_1+e_2+e_3=0$, the author proves by matrix-theoretical methods that the required law is of the form

$$\mathbf{E}=f(J_2, J_3^2)\{p(J_2, J_3^2)J_3\mathbf{T}+q(J_2, J_3^2)\mathbf{S}\}.$$

Here $J_2=\frac{1}{2}(s_1^2+s_2^2+s_3^2)$, $J_3=\frac{1}{3}(s_1^3+s_2^3+s_3^3)$, f is an arbitrary function and p and q are polynomials homogeneous in the s_n, the degree of p being lower by 4 than that of q; $\mathbf{E}$ is the strain tensor, $\mathbf{S}$ the stress deviation tensor and $\mathbf{T}$ a tensor defined by $\mathbf{S}^2-\frac{2}{3}J_2\mathbf{I}$.

It is shown that a special case of the author's stress-strain relation is sufficient to represent experimental results by G. I. Taylor and H. Quinney [Philos. Trans. Roy. Soc. London. Ser. A. 230, 323–362 (1931)]. It is also shown that stress-strain relations suggested by W. R. Bailey [Proc. Inst. Mech. Eng. 131, 131–269 (1935)] are special cases of the author's result. *E. Reissner* (Cambridge, Mass.).

Pizzetti, Giulio. Contributo allo studio dei sistemi in cui si verificano anche deformazioni non elastiche. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. 78, 49–59 (1943). [MF 16240]

Following ideas of G. Colonnetti [see, for instance, Pont. Acad. Sci. Comment. 2, 439–514 (1938)], the author determines the residual stresses in a drawn wire, making plausible assumptions concerning the distribution of the permanent axial and radial strains over the cross section. *W. Prager* (Providence, R. I.).

van Iterson, F. K. Th. Contributions to the theory of plasticity. Nederl. Akad. Wetensch. Verslagen, Afd. Natuurkunde 52, 5–11 (1943). (Dutch. German, English and French summaries) [MF 15771]

It is pointed out that even in the so-called two-dimensional problems of plasticity the yield condition involves all three principal stresses. The author suggests the hypothesis that in these two-dimensional problems the third principal stress always equals one of the two other principal stresses. *W. Prager* (Providence, R. I.).

Finzi, Bruno. Il problema ristretto tridimensionale nella teoria della plasticità. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. 76, 222–238 (1941). [MF 16266]

The author establishes the compatibility equations for the components of the stress deviation in a three-dimensional plastic continuum of the Saint Venant-von Mises type in the absence of body forces. *W. Prager.*

Vening Meinesz, F. A. Equations for elastic solids in spherical coordinates, including the case that the temperature is not constant in space and that a gravitational force is working in the sense of the radius. Solution of these equations, also applicable to viscous fluids, for the general problem that the radial components P_r and V_r of the mass-forces and of the elastic displacements resp. the velocities, as well as the normal components σ_r of the stresses on the spheres and the temperature θ are functions of ρ multiplied by the same spherical harmonic, while the components P and V on the spheres of the mass-forces and of the elastic displacements resp. the velocities, and the shearing-stresses τ on the spheres are functions of ρ multiplied by gradients of this spherical harmonic. Nederl. Akad. Wetensch., Proc. 48, 469–486 (1945). [MF 16587]

Seth, B. R. Finite strain in aelotropic elastic bodies. II. Bull. Calcutta Math. Soc. 38, 39–44 (1946). [MF 16972]

For part I see the same Bull. 37, 62–68 (1945); these Rev. 7, 143.

Morris, Rosa M. Some general solutions of St. Venant's torsion and flexure problem. II. Proc. London Math. Soc. (2) 49, 1–18 (1945). [MF 13706]

In part I [same Proc. (2) 46, 81–98 (1940); these Rev. 1, 189] the author considered the torsion and flexure of uniform beams the cross sections of which could be mapped by a transformation of the type $z=\sum_{n=0}^{\infty}a_ne^{ni\zeta}$ from the z-plane onto the strip $0<\xi<2\pi$, $0<\eta<\infty$ in the ζ-plane, $\zeta=\xi+i\eta$. The problem was then solved through the determination of six functions analytic within the strip and satisfying certain

boundary conditions. In the present paper the additional case is considered when the transformation is $z=\sum_{n=-1}^{\infty} a_n e^{n\zeta}$. This approach yields the solution of the torsion and flexure problems for additional cross sections. *G. E. Hay.*

Pizzetti, Giulio. I solidi a grande curvatura in campo elasto-plastico. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **78**, 31–48 (1943). [MF 16239]

The paper is concerned with the elastic-plastic flexure of a bar with pronounced initial curvature. It is assumed that the normal cross sections remain plane and normal to the deformed center line and that the material does not exhibit strain hardening. *W. Prager* (Providence, R. I.).

de Beer, C. On the stresses in prismatic shafts with cross section bounded by two pairs of orthogonal circular arcs. Nederl. Akad. Wetensch., Proc. **48**, 301–315 (1945). (Dutch) [MF 15921]

Sikorski, G. Application des fonctions elliptiques à l'étude de quelques cas de la flexion de tiges. Bull. École Polytech. Bucarest [Bul. Politehn. Bucureşti] **13**, 336–345 (1942). [MF 13570]

A class of beams is considered which includes the I-beam and the rectangular beam. The loading is transverse, the width of the cross section is uniform and the depth of the cross section is varied along the beam so that the maximum fiber stress is constant. It is found that, when the loading is linear or parabolic, the deflection of the central line can be found in terms of elliptic functions. The corresponding problem when the beam has a circular cross section of varying radius is also considered. [A broader treatment of problems involving beams of uniform strength has been given recently by I. Opatowski [Quart. Appl. Math. **3**, 76–81 (1945); J. Appl. Mech. **12**, A-156–A-158 (1945); these Rev. **6**, 196; **7**, 42].] *G. E. Hay* (Ann Arbor, Mich.).

Durant, N. J. An application of the method of finite difference equations to a problem of bending moments. (Continuous beam of N equal spans under uniform loading.) Philos. Mag. (7) **35**, 848–850 (1944). [MF 12957]

The author considers a uniform beam with N equal spans. The $N+1$ supports are all at the same level and the intensity of loading is uniform. It is desired to find the bending moment at each support, since a knowledge of this permits a determination of the bending moment at intermediate points by means of simple statics. Clapeyron's theorem of three moments is applied, yielding a set of relations each one of which involves linearly the bending moment at three adjacent supports. It is possible to solve these directly for the required bending moments, but with much labor. However, by considering the relations as difference equations, the author is able to solve them explicitly with great ease. *G. E. Hay* (Ann Arbor, Mich.).

Krall, G. Pressioni critiche e sforzi in un involucro cilindrico rinforzato con nervature solidali communque intervallate ed eventualmente impresse. Ann. Mat. Pura Appl. (4) **23**, 241–289 (1943). [MF 16617]

Saibel, Edward. Buckling loads of beams or plates on continuous supports. J. Aeronaut. Sci. **11**, 399–403 (1944); correction, **12**, 251. [MF 11254]

The paper's aim is to express the buckling modes of a beam or plate with intermediate supports as a combination of the known buckling modes without the intermediate supports. The coefficients of the series representing the unknown modes are to be determined by the energy method, in which the intermediate rigid supports are introduced by means of the method of the Lagrangian multiplier. [Cf. the following review.] *A. Weinstein* (Pittsburgh, Pa.).

Reissner, Eric. Buckling of plates with intermediate rigid supports. J. Aeronaut. Sci. **12**, 375–377 (1945). [MF 12695]

The author formulates his objections to the use of the multiplier method by E. Saibel [see the preceding review] as far as two-dimensional problems are concerned. The paper gives a sketch of a natural extension of the variational method of A. Weinstein [J. London Math. Soc. **10**, 184–192 (1935); Étude des spectres des équations aux dérivées partielles de la théorie des plaques élastiques, Mémor. Sci. Math., no. 88, Gauthier-Villars, Paris, 1937] to the case of plates with intermediate rigid support. The aim is to obtain lower bounds for the buckling load. A development in series [see E. Trefftz, Z. Angew. Math. Mech. **15**, 339–344 (1935); **16**, 64 (1936)] is used for rectangular plates. The difference between Saibel's and the author's approach may be characterized by saying that the former assumes the constraints accurately and intends to solve the problem approximately, while in the present paper constraints are assumed which approximate the real constraints and the resultant problem can be solved rigorously.
A. Weinstein (Pittsburgh, Pa.).

Reissner, Eric. Stresses and small displacements of shallow spherical shells. I. J. Math. Phys. Mass. Inst. Tech. **25**, 80–85 (1946). [MF 16151]

For thin spherical shell segments of small height and loaded to small displacements, the author obtains a set of simultaneous equations:
$$\nabla^2\nabla^2 F-(tE/R)\nabla^2 w=-(1-\nu)\nabla^2\Omega,$$
$$D\nabla^2\nabla^2 w+R^{-1}\nabla^2 F=p-2\Omega/R.$$
The coordinates are r and θ, where r is the distance from the apex of the shell measured in a plane parallel to the base plane, R the radius of the sphere, E the Young's modulus, t the thickness, D the flexural rigidity $Et^3/12(1-\nu^2)$, p the normal load, Ω the load potential for the components p_r and p_θ of load intensity on the meridional circumference defined by $p_r=-\partial\Omega/\partial r$, $p_\theta=-r^{-1}\partial\Omega/\partial\theta$, w the normal displacement and F the stress function for the components N_{rr}, $N_{\theta\theta}$, $N_{r\theta}$ in the surface of the shell, defined by
$$N_{rr}=r^{-1}\partial F/\partial r+r^{-2}\partial^2 F/\partial\theta^2+\Omega,$$
$$N_{\theta\theta}=\partial^2 F/\partial r^2+\Omega,\quad N_{r\theta}=-\partial(r^{-1}\partial F/\partial\theta)/\partial r.$$
H. S. Tsien (Cambridge, Mass.).

Yuan, Shao Wen. Thin cylindrical shells subjected to concentrated loads. Quart. Appl. Math. **4**, 13–26 (1946). [MF 15943]

The author first shows that the equation for small radial deflection w of a thin cylindrical shell of uniform thickness h is
$$\nabla^8 w+12a^{-2}h^{-2}(1-\nu^2)\partial^4 w/\partial x^4=D^{-1}\nabla^4 q,$$
where a is the radius, x the coordinate in the direction of the axis of the cylinder, ν the Poisson ratio, D the flexural rigidity $Eh^3/12(1-\nu^2)$ and q the lateral loading per unit area of the surface. The case of an infinitely long cylinder loaded by two diametrically opposite concentrated forces is calculated by using Fourier series in the circumferential direction and a Fourier integral in the axial direction. The result is expressed as a simple series. Numerical values for the maximum deflection under the load, the deflection w

along the generator through the point of loading and the contour lines of deflection w are given in various graphs. This solution is then used to calculate the deflection of a cylinder of finite length loaded by two diametrically opposite concentrated forces at the center section. The author indicates that cylinders loaded by concentrated torques can also be easily calculated by means of the given solution.

H. S. Tsien (Cambridge, Mass.).

Pell, William H. Thermal deflections of anisotropic thin plates. Quart. Appl. Math. **4**, 27–44 (1946). [MF 15944]

The paper contains a derivation of the differential equation governing the deflection of a thin anisotropic plate whose plane of elastic symmetry is parallel to the faces of the plate. The plate is subjected to a temperature distribution of the form $T(x, y, z) = T_0(x, y) + zT_1(x, y)$, where the neutral plane of the plate coincides with the xy-plane and T_0 and T_1 are fairly general functions of x and y. The paper makes use of conformal mapping to obtain the general solution for the stress functions with rather general boundary conditions and when $T_0(x, y)$ is a polynomial in x and y. The author obtains an explicit solution for the problem of thermal deflection of an isotropic circular plate with clamped and simply supported edges, when the temperature distribution is a function of the radius. For the case of an isotropic plate the author's results specialize to those given in A. Nadai's "Elastische Platten" [Springer, Berlin, 1925].

I. S. Sokolnikoff (Los Angeles, Calif.).

Zanaboni, Osvaldo. Lastra rettangolare con forze e coppie distribuite su rette, oppure sulla intera sua superficie. Ann. Mat. Pura Appl. (4) **20**, 195–210 (1941). [MF 16601]

Continuation of a paper in the same Ann. (4) **19**, 107–124 (1940); these Rev. **2**, 175.

Zanaboni, Osvaldo. Equazioni di equilibrio e di congruenza, indefinite ed ai limiti, delle lastre semielastiche a doppia curvatura. Ann. Mat. Pura Appl. (4) **23**, 215–239 (1944). [MF 16616]

Ghosh, S. A note on average stresses in a plate. Bull. Calcutta Math. Soc. **38**, 10–20 (1946). [MF 16968]

Ghosh, S. On the concept of generalized plane stress. Bull. Calcutta Math. Soc. **38**, 45–56 (1946). [MF 16973]

Schulz, K. J. On the state of stress in perforated strips and plates. V. Nederl. Akad. Wetensch., Proc. **48**, 282–291 (1945). [MF 15917]

Schulz, K. J. On the state of stress in perforated strips and plates. VI. Nederl. Akad. Wetensch., Proc. **48**, 292–300 (1 plate) (1945). [MF 15918]

These papers are the concluding sections of a series. The methods previously used [parts I–IV, same Proc. **45**, 233–239, 341–346, 457–464, 524–532 (1942); these Rev. **5**, 250] are applied to the problem of the infinite strip, perforated with a single row of holes, in pure bending, and that of the semi-infinite plate with a row of holes parallel to its edge. It is assumed that the boundary stresses of each hole give rise to a resultant force P perpendicular to and directed towards the straight boundary of the plate.

H. W. March (Madison, Wis.).

Sneddon, Ian N. Boussinesq's problem for a flat-ended cylinder. Proc. Cambridge Philos. Soc. **42**, 29–39 (1946). [MF 14409]

In an earlier paper by the author and J. W. Harding [same Proc. **41**, 16–26 (1945); these Rev. **6**, 251] a theory was developed for the elastic problem arising when a rigid punch is pushed against a semi-infinite elastic body. One type of punch considered was a flat-ended circular cylinder with generators perpendicular to the plane boundary of the elastic body. In this case the solution was obtained but an investigation of the actual stress distribution was not carried out. In the present paper this question is investigated by numerical computation and the results are illustrated in tables and graphs.

G. E. Hay (Ann Arbor, Mich.).

Hayasi, Hirosi. On deformation of the earth's surface under the influence of a travelling disturbance. I. Proc. Phys.-Math. Soc. Japan (3) **24**, 533–548 (1942). [MF 15033]

Hayasi, Hirosi. On deformation of the earth's surface under the influence of a travelling disturbance. II. Proc. Phys.-Math. Soc. Japan (3) **24**, 800–808 (1942). [MF 15044]

Hayasi, Hirosi. On deformation of the earth's surface under the influence of a travelling disturbance. III. Proc. Phys.-Math. Soc. Japan (3) **24**, 915–922 (1942). [MF 15047]

Hayasi, Hirosi. On deformation of the earth's surface under the influence of a travelling disturbance. IV. Proc. Phys.-Math. Soc. Japan (3) **25**, 47–56 (1943). [MF 15049]

Hayasi, Hirosi. On deformation of the earth's surface under the influence of a travelling disturbance. V. Proc. Phys.-Math. Soc. Japan (3) **25**, 198–206 (1943). [MF 15054]

Hayasi, Hirosi. On deformation of the earth's surface under the influence of a travelling disturbance. VI. Proc. Phys.-Math. Soc. Japan (3) **25**, 391–395 (1943). [MF 15060]

Hayasi, Hirosi. On deformation of the earth's surface under the influence of a travelling disturbance. VII. Proc. Phys.-Math. Soc. Japan (3) **25**, 468–480 (1943). [MF 15065]

Hayasi, Hirosi. On deformation of the earth's surface under the influence of a travelling disturbance. VIII. Proc. Phys.-Math. Soc. Japan (3) **25**, 648–658 (1943). [MF 15076]

Let $\mathbf{D}$ be the displacement vector. The well-known first order approximation for a homogeneous isotropic elastic solid, neglecting body forces, leads to the following relations. Write $\mathbf{D} = \nabla\phi + \nabla\times\mathbf{A}$ and $\Box_a = \Delta - a^{-2}\partial^2/\partial t^2$. Then $\Delta\phi = 0$, $\Box_b(\nabla\nabla\cdot - \Delta)\mathbf{A} = 0$. The author shows that essentially $\mathbf{D}$ is determined by the special case (1) $\mathbf{D} = \nabla\phi + \mathbf{D}'$, $\Box_a\phi = 0$, where (2) $\nabla\cdot\mathbf{D}' = 0$, $\Box_b\mathbf{D}' = 0$ and (3) $\nabla\cdot\mathbf{A} = 0$, $\Box_b\mathbf{A} = 0$. He treats the problem of the half space $z \leqq 0$ subject to the stresses (4) $Z_z|_{z=0} = f(x, y) = C$ or 0, $C > 0$, depending on whether (x, y) is interior or exterior to a circle of radius a, and all other external stresses at $t = 0$ are 0. The Cauchy-Fourier integral represents Q as

$$(5)\qquad \iint Q(\alpha, \beta)(\exp(-\gamma z + i((\alpha x - \alpha vt) + \beta y)))d\alpha d\beta,$$

where $Q = \phi$ or $\mathbf{A}$ and γ, α, β are quadratically related in view of (1) or (3). Physically (5) represents the disturbance as the superposition of travelling waves of velocity v. The usual stress-strain relations yield equivalents of (4) in the strains and thus, using (5) with $x + iy = r \exp i\theta$ and $r = \omega a$,

$$(6)\qquad D_z|_{z=0,\, t=0} \sim \int_0^\infty (J_1(s)/s)ds \int_{-\pi}^{\pi} h(\psi) \exp(i\omega s \cos(\psi - \theta))d\psi.$$

The expression for D_z (D_y) at $z=0$, $t=0$ differs only by a slight change in $h(\psi)$. Write $\epsilon=v^2/b^2$, $\nu=b^2/a^2$, (7) $\tau=\epsilon\cos^2\psi$. Then $h(\psi)=F(\tau)=\tau(1-\nu\tau)^{\frac{1}{2}}/(4((1-\nu\tau)(1-\tau))^{\frac{1}{2}}-(2-\tau)^2)$. There are branch points at $\tau=1, \nu^{-1}, \infty$ and poles at $\tau_0, \tau_1, \bar{\tau}_1$, where τ_0 is real. From (7) each singular point of $F(\tau)$ corresponds to $\pm\psi$, $\pm(\pi-\psi)$. The odd-numbered papers evaluate (6), the even-numbered papers evaluate $D_z|_{z=0,t=0}$, for each of the following cases: (I) $\epsilon<\tau_0$, (II) $\tau_0<\epsilon<1$, (III) $1<\epsilon<\nu^{-1}$, (IV) $\nu^{-1}<\epsilon$. For (I), for instance, one expands $\exp(i\omega s\cos(\psi-\theta))$ in a series of products $J_n(\omega s)\cos n(\psi-\theta)$ (essentially Laurent's series, though mathematical physicists usually refer to complicated derivations). Then $F(\tau)$ is expanded in a Taylor series and the iterated integral (6) is computed termwise after using (7). Essentially the same method is used in the other papers. The modifications arise in taking cognizance of the pole τ_0 and the fact that $h(\psi)$ may not be real on the integration range. These complications are taken care of by deforming the contour in the τ-Riemann 3-sheeted surface. On using (7) one gets back to ψ integration and essentially the result is that in (I) except for added residue terms. *D. G. Bourgin* (Urbana, Ill.).

Fu, C. Y. Studies on seismic waves. I. Reflection and refraction of plane waves. Geophysics 11, 1–9 (1946).
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It is assumed that the constrained oscillations are caused by uniformly distributed periodic disturbing forces, applied on the contour of the circle of cross section of the cylinder in the plane $z=0$. In cases of free oscillations it is established that the velocity of propagation of the free waves along the cylinder varies from the velocity of the Rayleigh wave to the velocity of the cross sectional wave. Resonance is investigated when the frequency of the disturbing force is equal to the frequency of the free oscillation, in case normal stress is applied to the contour. *From the author's summary.*

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This is a continuation of the author's previous studies of the natural frequencies of torsional vibration of a system consisting of n masses mounted on a shaft [same C. R. 217, 285–287, 421–422 (1943); these Rev. 6, 84, 140]. The note gives, without detailed proofs, some theorems concerning the effects of modifications of the system upon the distribution of the natural frequencies. *L. A. MacColl.*

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Acta Salmanticensia.

Vol. 1, no. 1 appeared in 1946. The journal is published by the University of Salamanca. There are to be several series; mathematical papers appear in the series entitled Ciencias: Sección de Matemáticas.

Annales Academiæ Scientiarum Fennicæ. Series A.

This series has been divided into five subseries, of which subseries I is Mathematica-Physica. Papers in the subseries appear as separate numbers, which are not collected into numbered volumes. The first number of the new series appeared in 1941; the last number of vol. 59 of the old series appeared in 1943. Vol. 60 of the old series is to be a general index. The main title is in Finnish: Suomalaisen Tideakatemian Toimituksia. Sarja A.

Fundamenta Mathematicae 33.

The first 114 pages of this volume were printed in 1939; some of the papers in these pages were reviewed in Mathematical Reviews from reprints. The remainder of the volume was printed and the complete volume issued in 1945.

Hungarica Acta Mathematica.

Vol. 1, no. 1 appeared in 1946. The journal is published in Budapest by the Hungarian Academy of Natural Sciences; numbers will appear at irregular intervals. Papers will be in English, French or German.

Intermédiaire des Recherches Mathématiques.

This quarterly is published in Paris. Vol. 1, no. 1, dated January 1945, was issued in September 1945. The following statement of its aims is taken from the journal. Buts: Aider les recherches mathématiques désintéressées; renseigner sur toute question mathématique, quel qu'en soit le niveau ou la spécialité; faciliter les contacts entre les chercheurs isolés, et indiquer les spécialistes; signaler les problèmes mathématiques non résolus et les sujets de recherches, même s'ils proviennent d'autres branches de la science; tenir au courant de l'actualité mathématique; collaborer aux réalizations mathématiques d'intérêt collectif; contribuer aux échanges internationaux. Questions: Recherches bibliographiques, problèmes embarrassants, idées que l'on n'a pas le temps d'exploiter soi-même, questions non résolues de differents recueils.

Matematisk Tidsskrift B. 1946.

This volume has been reprinted as part II of a Festskrift in honor of N. E. Nørlund. For part I, see these Rev. 7, 104.

Riveon Lematematika.

Vol. 1, no. 1 appeared in June, 1946. This quarterly, entirely in Hebrew, is published in Jerusalem.

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